Exercise Sheet 5

1. The goal of this exercise is to compute the "probability" that two integers m and n, both $\leq x$, are coprime.

1. Let $x \geq 1$ be a real number. Show that

$$|\{(m,n) \mid 1 \leq m,n \leq x \ \ (m,n)=1\}| = \sum_{d \leq x} \mu(d) \sum_{\substack{m,n \leq x \\ d \mid (m,n)}} 1,$$

where (m,n) denotes the gcd of m and n.

2. Deduce that

$$|\{(m,n) \mid 1 \leq m,n \leq x \ \ (m,n)=1\}| = \frac{6}{\pi^2}x^2 + O(x \log x)$$

for $x \geq 2$.

2. Let $f \geq 0$ be an arithmetic function.

1. Suppose that for every integer $k \geq 1$, the Dirichlet series

$$\sum_{n \geq 1} \frac{f(n)^k}{n^s}$$

for f^k converges for $\mathrm{Re}(s) > 1$. Prove then that for any $\epsilon > 0$, we have $f(n) \ll n^{\epsilon}$ for $n \geq 1$.

2. Deduce that, for all $\epsilon > 0$, the divisor function d satisfies $d(n) \ll n^{\epsilon}$ for all $n \geq 1$.

In the remainder of this exercise, we give a different proof of the last statement (which can be adapted to other functions).

3. Let $\epsilon > 0$ be given. Prove that there exists a real number P, depending only on ϵ, such that

$$d(p^v) \leq p^{v\epsilon}$$

for all $p \geq P$ and all integers $v \geq 1$.

4. Deduce that for all $\epsilon > 0$, the divisor function d satisfies $d(n) \ll n^{\epsilon}$ for all $n \geq 1$.

3. Let K be a number field. Let $r_K(n)$ be the arithmetic function defined by

$$r_K(n) = |\{\boldsymbol{n} \subset \mathbb{Z}_K \mid |\boldsymbol{n}| = n\}|$$

for all integers $n \geq 1$ (number of integral ideals of norm n).

1. Show that $r_K(n)$ is well-defined.
2. Show that r_K is a multiplicative function.
3. Let $k = [K : \mathbb{Q}]$. Show that for p prime and $v \geq 1$, we have

$$r_K(p^v) \leq |\{(a_1, \ldots, a_k) \mid a_i \geq 0 \text{ and } \sum_i a_i = v\}| \leq (v+1)^k.$$

4. Deduce that for all $\epsilon > 0$, we have the bound $r_K(n) \ll n^\epsilon$ for all $n \geq 1$. (Hint: use the previous exercise.)

4. Let f be an arithmetic function, and suppose that for every prime number p, there exist complex numbers α_p and β_p such that $\alpha_p \beta_p = 1$ and

$$\sum_{n \geq 1} f(n) n^{-s} = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}$$

for $\mathrm{Re}(s)$ large enough.

1. Show that for all primes p and all integers $v \geq 0$, we have

$$f(p^v) = \sum_{j=0}^{v} \alpha_p^j \beta_p^{v-j}.$$

2. Assume that, for all $\epsilon > 0$, we have $f(n) \ll n^\epsilon$ for $n \geq 1$. Let p be a prime number. Show that the power series

$$\sum_{v \geq 0} f(p^v) X^v$$

has radius of convergence ≥ 1, and deduce that $|\alpha_p| = |\beta_p| = 1$.

3. Conclude that, under the assumption of the previous question, we have $|f(n)| \leq d(n)$ for all $n \geq 1$.

5. We recall that $\varphi(n) = |(\mathbb{Z})/n\mathbb{Z})^\times|$ for all $n \geq 1$.

1. Prove that

$$\varphi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$$

for all $n \geq 1$.

2. Prove that
$$\sum_{n\leq x}\varphi(n)=\frac{3}{\pi^2}x^2+O(x\log x)$$
for $x\geq 1$.
3. Prove that
$$\varphi(n)=n\prod_{p|n}\Big(1-\frac{1}{p}\Big),$$
and deduce that $n/\varphi(n)=O(\log n)$ for $n\geq 2$. (Hint: bound it above by $\zeta(2)\sum_{d\leq n}1/d$.)
4. Deduce from problem (5.1) that the function $e(n)=|\{m\geq 1 \mid \varphi(m)=n\}|$ is a well-defined arithmetic function. Show that $\varphi(n)$ is even for all $n\geq 3$, and deduce that the function e is not multiplicative.
5. Prove that the Dirichlet series
$$F(s)=\sum_{n\geq 1}\frac{e(n)}{n^s}=\sum_{m\geq 1}\frac{1}{\varphi(m)^s}$$
converges absolutely for $\mathrm{Re}(s)>1$ and that we have in this region an equality
$$F(s)=\zeta(s)R(s)$$
where R is a function defined by an infinite product over primes which is holomorphic in the half-plane defined by $\mathrm{Re}(s)>0$. Does the existence of this factorization contradict the fact that e is not multiplicative?
6. Deduce that F has analytic continuation to the region $\mathrm{Re}(s)>0$ with a unique simple pole at $s=1$ with residue
$$r=\frac{\pi^2}{6}\prod_p\Big(1+\frac{1}{p^3}\Big).$$

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