Exercise Sheet 6

1. Using summation by parts, and assuming the prime number theorem in the form

$$\pi(x) = \frac{x}{\log x} + O\Big(\frac{x}{(\log x)^2}\Big)$$

for $x \geq 2$ (where $\pi(x)$ is the number of primes $p \leq x$), prove asymptotic formulas (as precise as you can) for

$$\sum_{p \leq x} p, \qquad \sum_{p \leq x} (\log p)^3.$$

2. For $n \geq 1$, we define $\omega(n)$ to be the number of prime factors of n, counted without multiplicity (so that $\omega(p^2) = 1$ for any prime number p, for instance).

For $x \geq 1$, define

$$\sigma_x = \sum_{p \leq x} \frac{1}{p}.$$

1. Using the formula
$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1)$$
for $x \geq 2$, proved in Exercise Sheet 1, Exercise 3, prove that
$$\sigma_x = \log \log x + O(1)$$
for $x \geq 3$.
2. Prove that
$$\sum_{n \leq x} \omega(n) = x \log \log x + O(x)$$
for $x \geq 3$.
3. Let $y = x^{1/2}$ and define
$$\omega'(n) = \sum_{\substack{p \mid n \\ p \leq y}} 1.$$
Prove that
$$\omega'(n) \leq \omega(n) \leq \omega'(n) + 1$$
for all integers $n \leq x$.

4. Prove that

$$\sum_{n\le x}\Big(\omega'(n)-\sum_{p\le y}\frac{1}{p}\Big)^2 = x\sigma_y + O(x).$$

(Hint: write

$$\omega'(n)-\sum_{p\le y}\frac{1}{p}=\sum_{p\le y}\Big(\delta_p(n)-\frac{1}{p}\Big),$$

where δ_p is the characteristic function of the integers divisible by p, and then expand the square and handle the various terms separately.)

5. Deduce that

$$\sum_{n\le x}(\omega(n)-\log\log x)^2 = x(\log\log x)+O(x(\sqrt{\log\log x})),$$

for $x\ge 3$. (This is a theorem of Hardy and Ramanujan.)

6. Suppose an integer n has size about 10^{100}, and that $\omega(n)=12$. Is that something remarkable?

3. For $x\ge 1$, define

$$M(x)=\sum_{n\le x}\mu(n),$$

where μ is the Möbius function.

1. Show that for $\mathrm{Re}(s)>1$, we have the equality

$$\frac{1}{\zeta(s)}=s\int_1^{+\infty}M(t)t^{-s-1}dt.$$

2. Deduce that, if the estimate

$$M(x)=O(x^\delta)$$

for $x\ge 2$ is true for a certain $\delta>0$, then $\zeta(s)\ne 0$ for all $s\in\mathbb{C}$ such that $\mathrm{Re}(s)>\delta$.

3. Similarly, prove that if the estimate

$$\sum_{n\le x}\Lambda(n)=x+O(x^\delta)$$

is valid for some $\delta>0$, then $\zeta(s)\ne 0$ for all $s\in\mathbb{C}$ such that $\mathrm{Re}(s)>\delta$.

4. The ternary divisor function d_3 is defined as the triple Dirichlet convolution $1\star 1\star 1$.

1. Compute the Dirichlet generating series $D(s)$ for d_3 and prove that it has meromorphic continuation to $\mathrm{Re}(s)>0$ with a triple pole at $s=1$.
2. Let $\epsilon>0$ be a real number. Prove that $d_3(n)\ll n^\epsilon$ for $n\ge 1$.

3. Let δ be a real number with $0 < \delta < 1$. Prove that

$$D_f(s) \ll (1 + |s|)^3$$

for $\operatorname{Re}(s) \geq \delta$ and $|\operatorname{Im}(s)| \geq 1$.

4. Let $\epsilon > 0$ be a real number. Using Mellin transform methods, prove that

$$\sum_{n \leq x} d_3(n) = x f(\log x) + O(x^{4/5+\epsilon})$$

for $x \geq 2$, where f is a polynomial of degree 2 with leading term $X^2/2$.

Due date: 16.12.2024