Solutions: Exercise Sheet 1

1. For a real number $x \geq 0$, we define

$$N(x) = |\{(a,b) \in \mathbb{Z}^2 \mid a^2 + b^2 \leq x\}|.$$

The goal of this exercise is to prove the estimate

$$N(x) = \pi x + O(x^{1/2}),$$

for $x \geq 1$ (recall that $f(x) = O(g(x))$ for $x \in X$ means that there exists a constant $c \geq 0$ such that $|f(x)| \leq cg(x)$ for all $x \in X$).

1. Show that
$$N(x) \leq \pi(\sqrt{x} + \sqrt{2})^2$$
for all $x \geq 0$. (Hint: interpret $N(x)$ as the area of a certain union of squares of side 1, contained in a suitable disc.)
Solution: The value of $N(x)$ is precisely the area of the union $\mathcal{N}$ of unit squares $[a, a+1) \times [b, b+1)$ where $(a,b) \in \mathbb{Z}^2$ and $a^2 + b^2 \leq x$, or equivalently $\|(a,b)\| \leq \sqrt{x}$ using the Euclidean metric. For each $(u,v) \in \mathcal{N}$, there exists $(a,b) \in \mathbb{Z}^2$ with $a^2 + b^2 \leq x$ and $(u,v) \in [a, a+1) \times [b, b+1)$. By the triangle inequality,
$$\begin{aligned}\|(u,v)\| &\leq \|(a,b)\| + \|(u-a, v-b)\| \\ &\leq \sqrt{x} + \|(1,1)\| \\ &\leq \sqrt{x} + \sqrt{2}.\end{aligned}$$
Thus $\mathcal{N}$ is contained in the disc of radius $\sqrt{x} + \sqrt{2}$, so that $N(x) = \text{Area}(\mathcal{N}) \leq \pi(\sqrt{x} + \sqrt{2})^2$.
2. Show that
$$N(x) \geq \pi(\sqrt{x} - \sqrt{2})^2$$
for all $x \geq 0$. (Hint: use a similar idea.)
Solution: As before, $N(x) = \text{Area}(\mathcal{N})$. Let (u,v) be any point with $\|(u,v)\| \leq \sqrt{x} - \sqrt{2}$, and let $(a,b) = (\lfloor u \rfloor, \lfloor v \rfloor)$. Then $(a,b) \in \mathbb{Z}^2$ and by the triangle inequality,
$$\begin{aligned}\|(a,b)\| &\leq \|(u,v)\| + \|(a-u, b-v)\| \\ &\leq \sqrt{x} - \sqrt{2} + \|(1,1)\| \\ &= \sqrt{x} - \sqrt{2} + \sqrt{2} = \sqrt{x},\end{aligned}$$
so $\mathcal{N}$ contains the square with bottom-left corner (a,b). The point (u,v) is contained in this square and thus in $\mathcal{N}$, so $\mathcal{N}$ contains the disc of radius $\sqrt{x} - \sqrt{2}$. Thus $N(x) = \text{Area}(\mathcal{N}) \geq \pi(\sqrt{x} - \sqrt{2})^2$.

3. Conclude.

Solution: The bounds of the previous two parts show that

$$-2\sqrt{2}\sqrt{x}+2 \leq N(x)-\pi x \leq 2\sqrt{2}\sqrt{x}+2.$$

Thus for example $|N(x)-\pi x| \leq 6\sqrt{x}$ for all $x \geq 1$, so that $N(x)=\pi x+O(x)$.

2. We order the primes in increasing order $(p_n)_{n\geq 1}$:

$$2=p_1<3=p_2<5=p_3<\cdots.$$

The goal of the exercise is to show that there exist positive constants c_1' and c_2' such that

$$c_1' n\log(n) \leq p_n \leq c_2' n\log(n) \tag{1}$$

for all $n \geq 1$.

1. Show that

$$\lim_{n\to+\infty} \frac{\log(n)}{\log(p_n)} = 1.$$

(Hint: observe first that $\pi(p_n)=n$, and then use Chebychev's estimate.)

Solution: By Chebychev's estimate, there exist positive constants c_1 and c_2 such that

$$c_1\frac{x}{\log x} \leq \pi(x) \leq c_2\frac{x}{\log x}.$$

Note that $\pi(p_n)=n$, so applying this estimate to $x=p_n$ and taking logarithms we get that

$$\log c_1+\log p_n-\log\log p_n \leq \log n \leq \log c_2+\log p_n-\log\log p_n.$$

Dividing by $\log p_n$, we get

$$1+\frac{\log c_1-\log\log p_n}{\log p_n} \leq \frac{\log n}{\log p_n} \leq 1+\frac{\log c_2-\log\log p_n}{\log p_n}.$$

Taking the limit as $n\to\infty$ (and thus as $p_n\to\infty$), we note that $\frac{\log c_1}{\log p_n}\to 0$, $\frac{\log c_2}{\log p_n}\to 0$, and $\frac{\log\log p_n}{\log p_n}\to 0$. Thus the limit of $\frac{\log n}{\log p_n}$ is bounded above and below by 1, so it must be 1.

2. Using again Chebychev's estimate, prove (1).

Solution: Since $\lim_{n\to\infty}\frac{\log n}{\log p_n}=1$, there exist constants d_1 and d_2 such that for all $n\geq 1$,

$$d_1 \leq \frac{\log n}{\log p_n} \leq d_2,$$

or equivalently

$$\frac{1}{d_2}\log n \leq \log p_n \leq \frac{1}{d_1}\log n. \tag{2}$$

Applying Chebychev's estimate to $x = p_n$, we get that

$$c_1 \frac{p_n}{\log p_n} \le n \le c_2 \frac{p_n}{\log p_n}$$
$$\Rightarrow c_1 p_n \le n \log p_n \le c_2 p_n.$$

The left inequality combined with (2) implies that $c_1 p_n \le n \log p_n \le \frac{1}{d_1} n \log n$, so that $p_n \le \frac{1}{c_1 d_1} n \log n$. By the same logic on the other side, $\frac{1}{c_2 d_2} n \log n \le p_n$. Taking $c_1' = \frac{1}{c_2 d_2}$ and $c_2' = \frac{1}{c_1 d_1}$ completes the proof of (1).

3. Prove that

$$\sum_{p \le x} \frac{1}{p} \to +\infty.$$

More precisely, how large can you get the partial sums

$$\sum_{p \le x} \frac{1}{p}$$

to be as $x \to +\infty$?

Solution: Write

$$\begin{aligned} \sum_{p \le x} \frac{1}{p} &= \sum_{n \le \pi(x)} \frac{1}{p_n} \\ &\ge \frac{1}{c_2'} \sum_{n \le \pi(x)} \frac{1}{n \log n}, \text{ by part (2)} \\ &\ge \frac{1}{c_2'} \int_2^{\pi(x)-1} \frac{1}{u \log u} \mathrm{d}u \\ &= \frac{1}{c_2'} (\log\log(\pi(x) - 1) - \log\log 2) \\ &\ge \frac{1}{2c_2'} \left(\log\log\left(\frac{c_1 x}{\log x}\right) - \log\log 2 \right), \end{aligned}$$

using that $\log\log(n-1) \ge \frac{1}{2} \log\log n$ for all $n \ge 1$ and Chebyshev's estimate. This in turn is larger than (for example) $\frac{1}{4c_2'} \log\log x$ for large enough x. Since $\log\log x \to \infty$, so does $\sum_{p \le x} \frac{1}{p}$.

If instead of using the upper bound $p_n \le c_2' n \log n$, we used the lower bound $p_n \ge c_1' n \log n$, and applied the same argument, we can show that $\sum_{p \le x} \frac{1}{p} \le d \log\log x$ for some constant d. Thus $\sum_{p \le x} \frac{1}{p}$ grows on the order of $\log\log x$.

3. We define the von Mangoldt function $\Lambda(n)$ for integers $n \ge 1$ by

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \ge 1 \\ 0 & \text{otherwise.} \end{cases}$$

1. Show that for $n \geq 1$, we have

$$\sum_{k=1}^{n} \Lambda(k) = O(n).$$

(Hint: split the sum into the sums over primes, squares of primes, etc, and use Chebychev's estimate.)

Solution: The von Mangoldt function $\Lambda(k)$ is supported on prime powers, so

$$\begin{aligned}\sum_{k=1}^{n} \Lambda(k) &= \sum_{p^j \leq n} \log p \\ &= \sum_{p \leq n} \log p \left\lfloor \frac{\log n}{\log p} \right\rfloor \\ &\leq \sum_{p \leq n} \log n \\ &= \pi(n) \log n \leq c_2 \frac{n}{\log n} \log n = c_2 n,\end{aligned}$$

where the last line follows from Chebychev's estimate. Thus $\sum_{k=1}^{n} \Lambda(k) = O(n)$.

2. Show that for any integer $n \geq 1$, we have

$$\sum_{k=1}^{n} \log(k) = \sum_{\substack{p,j \\ p^j \leq n}} \left\lfloor \frac{n}{p^j} \right\rfloor \log(p) = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor \Lambda(k).$$

Solution: Writing $n! = \prod_p p^{v_p(n)}$ and recalling the identity that

$$v_p(n) = \sum_{p^j \leq n} \left\lfloor \frac{n}{p^j} \right\rfloor,$$

we have

$$\begin{aligned}\sum_{k=1}^{n} \log k &= \log(n) \\ &= \log \left(\prod_p p^{v_p(n)} \right) \\ &= \sum_{p \leq n} v_p(n) \log p \\ &= \sum_{p^j \leq n} \left\lfloor \frac{n}{p^j} \right\rfloor \log p,\end{aligned}$$

as desired. For the second equality, note that Λ is supported on prime powers and is precisely $\log p$ when k is a power of p, so this follows by the definition of Λ.

3. Show that

$$\sum_{k=1}^{n} \log(k) = n\log(n) + O(n)$$

for $n \geq 1$. (Hint: compare the sum with an integral.)

Solution: We have

$$\begin{aligned}
\int_2^n \log(u)\mathrm{d}u \leq \sum_{k=1}^{n} \log k &\leq \int_1^n \log(u)\mathrm{d}u + \log(n+1) \\
\Rightarrow (u\log(u) - u|_2^n \leq \sum_{k=1}^{n} \log k &\leq (u\log(u) - u|_1^n + \log(n+1) \\
\Rightarrow n\log n - n - 2\log 2 + 2 \leq \sum_{k=1}^{n} \log k &\leq n\log n - n + 1 + \log(n+1).
\end{aligned}$$

The upper and lower bounds combined imply that $|\sum_{k=1}^{n} \log k - n\log n| = O(n)$, as desired.

4. Deduce that

$$\sum_{k=1}^{n} \frac{\Lambda(k)}{k} = \log(n) + O(1)$$

for $n \geq 1$, and that

$$\sum_{p \leq n} \frac{\log(p)}{p} = \log(n) + O(1).$$

(Hint: for the first, combine (1), (2) and (3) and $0 \leq x - \lfloor x \rfloor \leq 1$; for the second, show that the contribution to the first sum of squares and higher powers of primes is bounded.)

Solution: Note first that

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor \Lambda(k) &= \frac{1}{n}\sum_{k=1}^{n} \log k, \text{ by part (2)} \\
&= \frac{1}{n}(n\log n + O(n)), \text{ by part (3)} \\
&= \log n + O(1).
\end{aligned}$$

The sum we want to evaluate is, however, slightly different. Instead, we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{\Lambda(k)}{k} &= \frac{1}{n}\sum_{k=1}^{n} \frac{n}{k}\Lambda(k) \\
&= \frac{1}{n}\sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor \Lambda(k) + \frac{1}{n}\sum_{k=1}^{n} \left(\frac{n}{k} - \left\lfloor \frac{n}{k} \right\rfloor\right) \Lambda(k) \\
&= \log n + O(1) + O\left(\frac{1}{n}\sum_{k=1}^{n} \Lambda(k)\right),
\end{aligned}$$

since $\frac{n}{k} - \lfloor \frac{n}{k} \rfloor \leq 1$. Applying part (1) to the final term completes the argument.
For the second sum, along with the first result for this question, it suffices to show that

$$\sum_{\substack{p^j \leq n \\ j \geq 2}} \frac{\log p}{p^j} = O(1).$$

Summing first over j via geometric series, we have

$$\begin{aligned} \sum_{p \leq n} \log p \sum_{j \geq 2}^{\log_p(n)} \frac{1}{p^j} &\leq \sum_{p \leq n} \log p \sum_{j \geq 2} \frac{1}{p^j} \\ &= \sum_{p \leq n} \log p \frac{1}{p(p-1)} \\ &\leq \sum_{k=1}^{\infty} \frac{\log k}{k^2}, \end{aligned}$$

noting that for each prime p, $\frac{\log p}{p(p-1)} \leq \frac{\log(p-1)}{(p-1)^2}$, and that we only increase the sum by further extending to all integers k. But the sum over k is convergent (as seen for example by comparing $\frac{\log k}{k^2}$ to $\frac{1}{k^{3/2}}$), so it is bounded by a constant and thus $O(1)$, as desired.

4. (Optional but recommended) Using a computer, make a table of sums of three and four squares, and try to make suitable guesses or conjectures concerning the numbers that appear, which ones don't, and how many times an integer n might be a sum of three or four squares.

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