Exercise Sheet 5

1. The goal of this exercise is to compute the "probability" that two integers m and n, both $\leq x$, are coprime.

1. Let $x \geq 1$ be a real number. Show that

$$|\{(m,n) \mid 1 \leq m,n \leq x \ \ (m,n)=1\}| = \sum_{d \leq x} \mu(d) \sum_{\substack{m,n \leq x \\ d \mid (m,n)}} 1,$$

where (m,n) denotes the gcd of m and n.

Solution: By the Möbius inversion formula,

$$\begin{aligned} |\{(m,n) \mid 1 \leq m,n \leq x \ \ (m,n)=1\}| &:= \sum_{\substack{m,n \leq x \\ (m,n)=1}} 1 \\ &= \sum_{m,n \leq x} \sum_{d \mid (m,n)} \mu(d). \end{aligned}$$

Swapping sums gives the answer.

2. Deduce that

$$|\{(m,n) \mid 1 \leq m,n \leq x \ \ (m,n)=1\}| = \frac{6}{\pi^2} x^2 + O(x \log x)$$

for $x \geq 2$.

Solution: Starting from the previous problem, we have

$$\begin{aligned} \sum_{d \leq x} \mu(d) \sum_{\substack{m,n \leq x \\ d \mid (m,n)}} 1 &= \sum_{d \leq x} \mu(d) \Big(\sum_{\substack{n \leq x \\ d \mid n}} 1 \Big)^2 \\ &= \sum_{d \leq x} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor^2 \\ &= \sum_{d \leq x} \mu(d) \left(\frac{x^2}{d^2} + O\left(\frac{x}{d}\right) \right) \\ &= x^2 \sum_{d \leq x} \frac{\mu(d)}{d^2} + O\left(x \sum_{d \leq x} \frac{1}{d} \right). \end{aligned}$$

By the discussion surrounding equation (3.10) in the notes, the first sum is $\frac{x^2}{\zeta(2)} + O(x^{3/2})$, say, whereas by for example bounding the sum by an integral, one can see that the second term is $O(x \log x)$. This, along with $\zeta(2) = \pi^2/6$, gives the desired estimate.

2. Let $f \geq 0$ be an arithmetic function.

1. Suppose that for every integer $k \geq 1$, the Dirichlet series
$$\sum_{n\geq 1} \frac{f(n)^k}{n^s}$$
for f^k converges for $\mathrm{Re}(s) > 1$. Prove then that for any $\epsilon > 0$, we have $f(n) \ll n^\epsilon$ for $n \geq 1$.
Solution: Note that in order for the Dirichlet series $\sum_{n\geq 1} \frac{f(n)^k}{n^s}$ to converge, we must have $\frac{f(n)^k}{n^s} \to 0$ as $n \to \infty$. Fix any $\epsilon > 0$ and let k be large enough that $2/k < \epsilon$. The Dirichlet series $\sum_{n\geq 1} \frac{f(n)^k}{n^2}$ converges, so
$$\lim_{n\to\infty} \frac{f(n)^k}{n^2} = 0.$$
This implies that for large enough n,
$$\frac{f(n)^k}{n^2} = \left(\frac{f(n)}{n^{2/k}}\right)^k \leq 1,$$
and thus for large enough n, $f(n)/n^{2/k} \leq 1$. Thus $f(n) \ll n^{2/k} \ll n^\epsilon$.
2. Deduce that, for all $\epsilon > 0$, the divisor function d satisfies $d(n) \ll n^\epsilon$ for all $n \geq 1$.
Solution:
We present here multiple solutions. First, the following argument makes use of Proposition 3.6.2 (2):
For the multiplicative function $f(n) = d(n)^k$, we have that $f(p) = 2^k$ for all primes p, and for $v \geq 2$ we have $f(p^v) = (v+1)^k = O(v^k)$. Thus we can apply Proposition 3.6.2, (2), where the real number $A \geq 0$ is given by k, $\delta > 0$ can be anything (say $\delta = 2$), and the "k" in the proposition is our 2^k, to get that
$$\sum_{n\geq 1} \frac{f(n)}{n^s} = \zeta(s)^{2^k} D_f^\sharp(s),$$
where $D_f^\sharp(s)$ is holomorphic for $\mathrm{Re}(s) > 1-\delta$. Since $\zeta(s)^{2^k}$ and $D_f^\sharp(s)$ both converge for $\mathrm{Re}(s) > 1$, so does $\sum_{n\geq 1} \frac{f(n)}{n^s}$.
Thus by problem (2.1), for all $\epsilon > 0$, $d(n) \ll n^\epsilon$ for all $n \geq 1$.
Second, the following solution makes use of an inductive bound in terms of $\zeta(s)$:
First note that for any k, for any prime power p^v,
$$\begin{aligned} d(p^v)^k * d(p^v)^k &= \sum_{0\leq j\leq v} d(p^j)^k d(p^{v-j})^k \\ &= \sum_{0\leq j\leq v} (j+1)^k (v-j+1)^k \\ &\geq \sum_{0\leq j\leq v} (v+1)^k \\ &= (v+1)^{k+1} = d(p^v)^{k+1}. \end{aligned}$$

Thus by induction, $d(p^v)^{k+1} \leq d(p^v)^{*2^k}$, where $f(n)^{*k}$ denotes the convolution of k copies of $f(n)$. By multiplicativity the same holds for all n, so that

$$\sum_{n=1}^{\infty} \frac{d(n)^k}{n^s} \leq \sum_{n=1}^{\infty} \frac{d(n)^{(*2^{k-1})}}{n^s} = \zeta(s)^{2^k}.$$

Since $\zeta(s)^{2^k}$ converges absolutely for $\mathrm{Re}(s) > 1$, so does $\sum_{n=1}^{\infty} \frac{d(n)^k}{n^s}$.

In the remainder of this exercise, we give a different proof of the last statement (which can be adapted to other functions).

3. Let $\epsilon > 0$ be given. Prove that there exists a real number P, depending only on ϵ, such that

$$d(p^v) \leq p^{v\epsilon}$$

for all $p \geq P$ and all integers $v \geq 1$.

Solution: Fix $\epsilon > 0$. For all $v \geq 1$, we have $d(p^v) = |\{p^a \mid 0 \leq a \leq v\}| = v + 1$. Let P be large enough that $p^\epsilon \geq 2$ whenever $p \geq P$. By single-variable calculus, $1 + x \leq 2^x$ for all $x \geq 1$. Thus for all $v \geq 1$,

$$v + 1 \leq 2^v \leq (p^\epsilon)^v = p^{v\epsilon}.$$

4. Deduce that for all $\epsilon > 0$, the divisor function d satisfies $d(n) \ll n^\epsilon$ for all $n \geq 1$.

Solution: Fix $\epsilon > 0$. By the previous part and multiplicativity, $d(n) \ll n^\epsilon$ for all n divisible only by primes $p \geq P$.

Let $p < P$ be a prime. Note that as $v \to \infty$, $d(p^v) = v + 1 = o(p^{v\epsilon})$, so there exists some constant $V_p \geq 1$ such that $v + 1 \leq p^{v\epsilon}$ for all $v \geq V_p$. Define

$$M_p = \max_{1 \leq v \leq V_p} \frac{v+1}{p^{v\epsilon}},$$

so that $v + 1 \leq M_p p^{v\epsilon}$ for all $v \geq 1$. Define further

$$M = \max_{p \leq P} M_p.$$

Since P depends only on ϵ and V_p, M_p depend only on P, M is finite and depends only on ϵ. By construction, for all p and all v,

$$d(p^v) \leq M p^{v\epsilon}.$$

Thus

$$\begin{aligned} d(n) = \prod_{p \mid n} d(p^{v_p(n)}) &= \prod_{p<P} d(p^{v_p(n)}) \prod_{p \geq P} d(p^{v_p(n)}) \\ &\leq \prod_{p<P} M p^{v_p(n)\epsilon} \prod_{p \geq P} p^{v_p(n)\epsilon} \\ &\leq M^P n^\epsilon. \end{aligned}$$

Thus $d(n) \ll n^\epsilon$.

3. Let K be a number field. Let $r_K(n)$ be the arithmetic function defined by

$$r_K(n) = |\{\boldsymbol{n} \subset \mathbb{Z}_K \mid |\boldsymbol{n}| = n\}|$$

for all integers $n \geq 1$ (number of integral ideals of norm n).

1. Show that $r_K(n)$ is well-defined.

   Solution: We need to show that for all integers $n \geq 1$, the number of integral ideals of norm n is finite. Since every ideal factors as a product of prime ideals, it suffices to show that for $n = p^k$, there are finitely many prime ideals of norm n.

   Fix an integral prime p; we will show that there are finitely many prime ideals $I \subset \mathbb{Z}_K$ such that $I \cap \mathbb{Z} = p\mathbb{Z}$, which suffices since the intersection of any prime ideal in $\mathbb{Z}_K$ with $\mathbb{Z}$ must remain prime. By Lemma 2.7.1 (1), these ideals are precisely those prime ideals appearing in the factorization

   $$p\mathbb{Z}_K = \mathbf{p}_1^{e_1} \cdots \mathbf{p}_g^{e_g},$$

   and each $\mathbf{p}_i$ has norm a power of p by Lemma 2.7.1 (2). By Lemma 2.7.1 (3), $g \leq [K : \mathbb{Q}]$, so g must be finite. This is exactly what we wanted to show.

2. Show that r_K is a multiplicative function.

   Solution: Recall that ideals in $\mathbb{Z}_K$ factor uniquely into prime ideals, and that all prime ideals in $\mathbb{Z}_K$ have norm that is a prime power. Let $m, n \geq 1$ be coprime integers. Then any ideal I of norm mn must factor uniquely as $I_m \cdot I_n$, where I_m has norm m and I_n has norm n. Thus

   $$\begin{aligned} r_K(mn) &= |\{I \subset \mathbb{Z}_K \mid |I| = mn\}| \\ &= |\{\boldsymbol{m}, \boldsymbol{n} \subset \mathbb{Z}_K \mid |\boldsymbol{m}| = m, |\boldsymbol{n}| = n\}| \\ &= |\{\boldsymbol{m} \subset \mathbb{Z}_K \mid |\boldsymbol{m}| = m\}| \cdot |\{\boldsymbol{n} \subset \mathbb{Z}_K \mid |\boldsymbol{n}| = n\}| \\ &= r_K(m) r_K(n), \end{aligned}$$

   so r_K is multiplicative.

3. Let $k = [K : \mathbb{Q}]$. Show that for p prime and $v \geq 1$, we have

   $$r_K(p^v) \leq |\{(a_1, \ldots, a_k) \mid a_i \geq 0 \text{ and } \sum_i a_i = v\}| \leq (v+1)^k.$$

   Solution: Write $p\mathbb{Z}_K = \mathbf{p}_1^{e_1} \cdots \mathbf{p}_g^{e_g}$, where as before $g \leq k$. The prime ideals $\mathbf{p}_1, \ldots, \mathbf{p}_g$ are precisely the set of prime ideals in $\mathbb{Z}_K$ whose norm is a power of p. Write $|\mathbf{p}_i| = p^{f_i}$ for all i and for $f_i \geq 1$. By unique factorization of prime ideals, any ideal I with $|I| = p^v$ can be written uniquely as $I = \mathbf{p}_1^{a_1} \cdots \mathbf{p}_g^{a_g}$, where $a_i \geq 0$ and, by taking norms on both sides,

   $$v = a_1 f_1 + \cdots + a_g f_g.$$

Thus

$$\begin{aligned}
r_K(p^v) &= |\{I \subset \mathbb{Z}_K \mid |I| = p^v\}| \\
&= |\{\mathbf{p}_1^{a_1} \cdots \mathbf{p}_g^{a_g} \subset \mathbb{Z}_K \mid v = a_1 f_1 + \cdots + a_g f_g\}| \\
&= |\{(a_1 f_1, \ldots, a_g f_g) \mid a_i f_i \geq 0 \text{ and } \sum_i a_i f_i = v\}| \\
&\leq |\{(a_1, \ldots, a_g) \mid a_i \geq 0 \text{ and } \sum_i a_i = v\}|,
\end{aligned}$$

where in the last step we have extended the set by allowing for all g-tuples $(a_1, \ldots, a_g)$ of nonnegative integers whose sum is v, instead of merely those where each $b_i = a_i f_i$ is a multiple of f_i. We obtain the desired result by noting that $g \leq k$, so we can extend the set further by considering k-tuples $(a_1, \ldots, a_k)$ rather than g-tuples.

For any $(a_1, \ldots, a_k)$ with $a_i \geq 0$ and $\sum_i a_i = v$, we must have $0 \leq a_i \leq v + 1$ for all i, so there are at most $v + 1$ choices for each a_i and thus $\leq (v + 1)^k$ elements total in this set.

4. Deduce that for all $\epsilon > 0$, we have the bound $r_K(n) \ll n^\epsilon$ for all $n \geq 1$. (Hint: use the previous exercise.)

   Solution: For a prime power p^v, we have $d(p^v) = v + 1$, since the factorizations of p^v are precisely $p^a p^b$ where $0 \leq a, b \leq v$ and $a + b = v$, of which there are $v + 1$. Thus $r_K(p^v) \leq d(p^v)^k$, and by multiplicativity for all n, $r_K(n) \leq d(n)^k$.

   By the previous question, for all $\varepsilon > 0$, $d(n) \ll n^{\epsilon/k}$, and thus $r_K(n) \ll n^\epsilon$.

4. Let f be an arithmetic function, and suppose that for every prime number p, there exist complex numbers α_p and β_p such that $\alpha_p \beta_p = 1$ and

$$\sum_{n \geq 1} f(n) n^{-s} = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}$$

for $\mathrm{Re}(s)$ large enough.

1. Show that for all primes p and all integers $v \geq 0$, we have

$$f(p^v) = \sum_{j=0}^{v} \alpha_p^j \beta_p^{v-j}.$$

   Solution: By geometric series expansion, for large enough $\mathrm{Re}(s)$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{f(n)}{n^s} &= \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1} \\
&= \prod_p \left(1 + \frac{\alpha_p}{p^s} + \frac{\alpha_p^2}{p^{2s}} + \cdots\right)\left(1 + \frac{\beta_p}{p^s} + \frac{\beta_p^2}{p^{2s}} + \cdots\right) \\
&= \prod_p \sum_{v=0}^{\infty} \frac{1}{p^{vs}} \sum_{j=0}^{v} \alpha_p^j \beta_p^{v-j},
\end{aligned}$$

where the inner sum is the coefficient of $\frac{1}{p^{vs}}$ when the α_p component and the β_p component are multiplied. Let $g(n)$ be the multiplicative function defined on prime powers p^v by $g(p^v) = \sum_{j=0}^{v} \alpha_p^j \beta_p^{v-j}$. Then by the above, $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ for all s with large enough $\mathrm{Re}(s)$.
By Lemma 3.5.2, therefore, $f = g$, as desired.

2. Assume that, for all $\epsilon > 0$, we have $f(n) \ll n^{\epsilon}$ for $n \geq 1$. Let p be a prime number. Show that the power series

$$\sum_{v \geq 0} f(p^v) X^v$$

has radius of convergence ≥ 1, and deduce that $|\alpha_p| = |\beta_p| = 1$.
Solution: Assume that for all $\epsilon > 0$, $f(n) \ll n^{\epsilon}$ for all $n \geq 1$. Let p be a prime number and fix $X \in \mathbb{C}$ with $|X| < 1$. Let $\epsilon > 0$ be small enough that $p^{\epsilon}|X| < 1$. Then

$$\begin{aligned} \sum_{v \geq 0} |f(p^v) X^v| &\ll \sum_{v \geq 0} p^{v\epsilon} |X|^v \\ &= \sum_{v \geq 0} (p^{\epsilon}|X|)^v, \end{aligned}$$

which is a power series with ratio less than 1 and therefore converges. Thus $\sum_{v \geq 0} f(p^v) X^v$ has radius of convergence ≥ 1.
Assume by contradiction that $|\alpha_p| \neq 1$. Since $\alpha_p = \frac{1}{\beta_p}$ we can assume without loss of generality that $|\alpha_p| < 1$ and $|\beta_p| > 1$. Then choosing $X = \alpha_p$ we have

$$\begin{aligned} f(p^v) X^v &= \alpha_p^v \sum_{j=0}^{v} \alpha_p^j \beta_p^{v-j} \\ &= \sum_{j=0}^{v} \alpha_p^j \beta_p^{-j}, \text{ since } \alpha_p \beta_p = \alpha_p^v \beta_p^v = 1 \\ &= \sum_{j=0}^{v} \beta_p^{-2j} \\ &= \frac{1 - \beta_p^{-2v-2}}{1 - \beta_p^{-2}}. \end{aligned}$$

Since $|\beta_p| > 1$, this approaches 1 (and not 0) as $v \to \infty$. But then $\sum_{v \geq 0} f(p^v) X^v$ cannot converge, which contradicts the radius of convergence being ≥ 1. Thus $|\alpha_p| = |\beta_p| = 1$.

3. Conclude that, under the assumption of the previous question, we have $|f(n)| \leq d(n)$ for all $n \geq 1$.
Solution: Assume that $f(n) \ll n^{\epsilon}$ for $n \geq 1$. Then $|\alpha_p| = |\beta_p| = 1$ by the previous question, so that for any prime power p^v,

$$|f(p^v)| = \left| \sum_{j=0}^{v} \alpha_p^j \beta_p^{v-j} \right| \leq \sum_{j=0}^{v} |\alpha_p|^j |\beta_p|^{v-j} = v + 1 = d(p^v),$$

so by multiplicativity $|f(n)| \le d(n)$ for all $n \ge 1$.

5. We recall that $\varphi(n) = |(\mathbb{Z})/n\mathbb{Z})^\times|$ for all $n \ge 1$.

1. Prove that
$$\varphi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$$
for all $n \ge 1$.
Solution: For $d|n$, $\varphi(d)$ is the number of integers $1 \le k \le d$ with $(k,d) = 1$. The $\{1 \le k \le d \mid (k,d) = 1\}$ is in bijection with the set $\{1 \le \ell \le n \mid (\ell, n) = n/d\}$, via the transformation $\ell = \frac{n}{d} \cdot k$. Thus
$$n = |\{1 \le \ell \le n\}| = \sum_{d|n} |\{1 \le \ell \le n \mid (\ell, n) = n/d\}| = \sum_{d|n} \varphi(d).$$
Then by Möbius inversion, $\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$.

2. Prove that
$$\sum_{n \le x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x)$$
for $x \ge 1$.
Solution: By the previous part,
$$\begin{aligned}
\sum_{n \le x} \varphi(n) &= \sum_{n \le x} \sum_{d|n} \mu(d) \frac{n}{d} \\
&= \sum_{d \le x} \mu(d) \sum_{\substack{n \le x \\ d|n}} \frac{n}{d} \\
&= \sum_{d \le x} \mu(d) \sum_{k \le x/d} k, \text{ where } n = dk \\
&= \sum_{d \le x} \mu(d) \frac{\lfloor x/d \rfloor (\lfloor x/d \rfloor + 1)}{2} \\
&= \sum_{d \le x} \mu(d) \left(\frac{x^2}{2d^2} + O(x/d) \right), \text{ since } \lfloor y \rfloor = y + O(1) \\
&= \frac{x^2}{2} \left(\frac{1}{\zeta(2)} + O(x^{-1}) \right) + O\left(x \sum_{d \le x} 1/d \right),
\end{aligned}$$
where the first term is evaluated once again using the discussion after equation (3.10) and the second term is $O(x \log x)$. Simplifying, and substituting $\zeta(2) = \pi^2/6$, gives the result.

3. Prove that

$$\varphi(n) = n\prod_{p|n}\Big(1-\frac{1}{p}\Big),$$

and deduce that $n/\varphi(n) = O(\log n)$ for $n \geq 2$. (Hint: bound it above by $\zeta(2)\sum_{d\leq n}\frac{1}{d}$.)

<u>Solution:</u> Since $\frac{\varphi(n)}{n} = \sum_{d|n}\frac{\mu(d)}{d}$, where $\frac{\mu(d)}{d}$ is multiplicative, $\frac{\varphi(n)}{n}$ must also be multiplicative. Thus it suffices to prove that for a prime power p^v, $\frac{\varphi(p^v)}{p^v} = 1-\frac{1}{p}$. For this we also use problem (5.1):

$$\frac{\varphi(p^v)}{p^v} = \sum_{d|p^v}\frac{\mu(d)}{d} = \sum_{a=0}^{v}\frac{\mu(p^a)}{p^a} = 1-\frac{1}{p},$$

since $\mu(p^a) = 0$ for $a \geq 2$.

We then have

$$\begin{aligned}
\prod_{p|n}\left(1-\frac{1}{p}\right)^{-1} &= \prod_{p|n}\left(1-\frac{1}{p}\right)^{-1}\left(1+\frac{1}{p}\right)^{-1}\left(1+\frac{1}{p}\right) \\
&= \prod_{p|n}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{p|n}\left(1+\frac{1}{p}\right) \\
&\leq \zeta(2)\sum_{d|n}\frac{\mu(d)^2}{d} \\
&\leq \zeta(2)\sum_{d\leq n}\frac{1}{d} \\
&\ll \log n,
\end{aligned}$$

as desired.

4. Deduce from problem (5.3) that the function $e(n) = |\{m \geq 1 \mid \varphi(m) = n\}|$ is a well-defined arithmetic function. Show that $\varphi(n)$ is even for all $n \geq 3$, and deduce that the function e is not multiplicative.

<u>Solution:</u> By problem (5.3), if $n = \varphi(m)$, then $\frac{m}{n} = O(\log m)$, and thus $n = \Omega(m/\log m)$. Thus $\log n = \Omega(\log m - \log\log m) = \Omega(\log m)$, so $\frac{m}{n} = O(\log n)$ and $m = O(n\log n)$.

This implies that for each n, there are at most $O(n\log n)$ values m for which $n = \varphi(m)$, and thus in particular finitely many. Thus $e(n)$ is a well-defined arithmetic function.

Assume first that $n \geq 3$ is odd. Then $\varphi(n) = n\prod_{p|n}\frac{(p-1)}{p}$. Since n is divisible by some odd prime $p \geq 3$, $\varphi(n)$ is divisible by $p-1$, and thus even. Now assume that $n \geq 3$ is even. Then

$$\varphi(n) = n\prod_{p|n}\frac{p-1}{p} = \frac{n}{2}\prod_{\substack{p|n\\ p\geq 3}}\frac{p-1}{p}.$$

Since n is even, $\frac{n}{2}$ is an integer. If n is divisible by an odd prime $p \geq 3$, then $(p-1)|\varphi(n)$, so $\varphi(n)$ is even. If not, then $n = 2^k$ for some $k \geq 2$, and $\varphi(n) = \frac{n}{2}$ is even.

Thus for all $n \geq 3$, $\varphi(n)$ is even (and note that $\varphi(1) = \varphi(2) = 1$). This implies that, for example, $e(3) = 0$, but $e(6) \geq 1$ since $\varphi(7) = 6$, so e cannot be multiplicative.

5. Prove that the Dirichlet series

$$F(s) = \sum_{n \geq 1} \frac{e(n)}{n^s} = \sum_{m \geq 1} \frac{1}{\varphi(m)^s}$$

converges absolutely for $\mathrm{Re}(s) > 1$ and that we have in this region an equality

$$F(s) = \zeta(s) R(s)$$

where R is a function defined by an infinite product over primes which is holomorphic in the half-plane defined by $\mathrm{Re}(s) > 0$. Does the existence of this factorization contradict the fact that e is not multiplicative?

Solution: For $\mathrm{Re}(s) > 1$,

$$\sum_{m \geq 1} \left| \frac{1}{\varphi(m)^s} \right| \ll \sum_{m \geq 1} \frac{(\log n)^{\mathrm{Re}(s)}}{n^{\mathrm{Re}(s)}},$$

which converges. Thus $F(s)$ converges absolutely for $\mathrm{Re}(s) > 1$. In this region,

$$\begin{aligned} F(s) = \sum_{m \geq 1} \frac{1}{\varphi(m)^s} &= \sum_{m \geq 1} \frac{1}{n^s \prod_{p|n} (1 - 1/p)^s} \\ &= \prod_p \left(1 + \left(1 - \frac{1}{p} \right)^{-s} \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots \right) \right) \\ &= \prod_p \left(1 + \left(1 - \frac{1}{p} \right)^{-s} \frac{1}{p^s} \left(1 - \frac{1}{p^s} \right)^{-1} \right) \\ &= \zeta(s) \prod_p \left(1 - \frac{1}{p^s} + \frac{1}{p^s} \left(1 - \frac{1}{p} \right)^{-s} \right). \end{aligned}$$

Define $R(s) = \prod_p \left(1 - \frac{1}{p^s} + \frac{1}{p^s} \left(1 - \frac{1}{p} \right)^{-s} \right)$. For $\mathrm{Re}(s) > 0$, we have $\left(1 - \frac{1}{p} \right)^{-s} = 1 + O(s/p)$, so that

$$R(s) = \prod_p \left(1 + O(sp^{-1-s}) \right),$$

which converges absolutely.

Note that the pth factor is a function of s, but is not a convergent series of the form $\sum_{v \geq 0} \frac{f(p^v)}{p^{vs}}$, so we get no contradiction.

6. Deduce that F has analytic continuation to the region $\mathrm{Re}(s) > 0$ with a unique simple pole at $s = 1$ with residue

$$r = \frac{\pi^2}{6} \prod_p \Big(1 + \frac{1}{p^3}\Big).$$

<u>Solution:</u> From the previous problem, we can analytically continue $F(s)$ to the region $\mathrm{Re}(s) > 0$ by definining $F(s) = \zeta(s)R(s)$ whenever $\mathrm{Re}(s) > 0$. Then $F(s)$ has poles only when $\zeta(s)$ has poles (since $R(s)$ is holomorphic in this region), and thus $F(s)$ has a unique simple pole at $s = 1$. At $s = 1$, the residue of $F(s)$ is

$$\begin{aligned}
\mathrm{Res}_{s=1}(F(s)) &= \lim_{s\to 1}(s-1)\zeta(s)R(s) \\
&= R(1) \\
&= \prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^{-1}\right) \\
&= \prod_p \left(1 - \frac{1}{p} + \frac{1}{p-1}\right) \\
&= \prod_p \frac{p^2 - p + 1}{p(p-1)} \\
&= \prod_p \left(1 - \frac{1}{p^2}\right)^{-1} \left(\frac{p^3 + 1}{p^3}\right) \\
&= \zeta(2) \prod_p \left(1 + \frac{1}{p^3}\right),
\end{aligned}$$

which gives the desired expression under the substitution $\zeta(2) = \frac{\pi^2}{6}$.

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