D-MATH
Prof. Dr. Emmanuel Kowalski

Number Theory I

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Exercise Sheet 6

1. Using summation by parts, and assuming the prime number theorem in the form

$$\pi(x) = \frac{x}{\log x} + O\Big(\frac{x}{(\log x)^2}\Big)$$

for $x \geq 2$ (where $\pi(x)$ is the number of primes $p \leq x$), prove asymptotic formulas (as precise as you can) for

$$\sum_{p \leq x} p, \qquad \sum_{p \leq x} (\log p)^3.$$

Solution: Using summation by parts, we have

$$\begin{aligned}\sum_{p \leq x} p &= x\pi(x) + O(1) - \int_2^x \pi(u)\mathrm{d}u \\ &= x\pi(x) - \int_2^x \left(\frac{u}{\log u} + O\left(\frac{u}{(\log u)^2}\right)\right)\mathrm{d}u + O(1) \\ &= x\pi(x) - \left(\frac{u^2}{2\log u}\bigg|_2^x + O\left(\int_2^x \frac{u}{(\log u)^2}\mathrm{d}u\right) + O(1),\end{aligned}$$

by integration by parts. Note that

$$\begin{aligned}\int_2^x \frac{u}{(\log u)^2}\mathrm{d}u \leq x\int_2^x \frac{1}{(\log u)^2}\mathrm{d}u \\ &= O\left(x\int_2^x \left(\frac{1}{(\log u)^2} - \frac{2}{(\log u)^3}\right)\mathrm{d}u\right) \\ &= O\left(x\left(\frac{u}{(\log u)^2}\bigg|_2^x\right)\right) = O\left(\frac{x^2}{(\log x)^2}\right).\end{aligned}$$

Thus we get

$$\sum_{p \leq x} p = x\pi(x) - \frac{x^2}{2\log x} + O\left(\frac{x^2}{(\log x)^2}\right) = \frac{x^2}{2\log x} + O\left(\frac{x^2}{(\log x)^2}\right).$$

We now turn to the quantity $\sum_{p \leq x} (\log p)^3$, where we have

$$\begin{aligned}\sum_{p \leq x} (\log p)^3 &= (\log x)^3\pi(x) + O(1) - \int_2^x 3(\log u)^2\pi(u)\frac{\mathrm{d}u}{u} \\ &= x(\log x)^2 + O(x\log x) - \int_2^x (3\log u + O(1))\,\mathrm{d}u \\ &= x(\log x)^2 + O(x\log x) - 3(u\log u|_2^x + \int_2^x \mathrm{d}u + O(x) \\ &= x(\log x)^2 + O(x\log x),\end{aligned}$$

where the third line follows from the second by integration by parts.

2. For $n \geq 1$, we define $\omega(n)$ to be the number of prime factors of n, counted without multiplicity (so that $\omega(p^2) = 1$ for any prime number p, for instance).

For $x \geq 1$, define
$$\sigma_x = \sum_{p \leq x} \frac{1}{p}.$$

1. Using the formula
$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1)$$
for $x \geq 2$, proved in Exercise Sheet 1, Exercise 3, prove that
$$\sigma_x = \log\log x + O(1)$$
for $x \geq 3$.

Solution: We first consider the contribution to $\sum_{n \leq x} \frac{\Lambda(n)}{n}$ from higher prime powers, which is given by
$$\sum_{k=2}^{\infty} \sum_{p \leq x^{1/k}} \frac{\log p}{p^k} \leq \sum_{p \leq x^{1/2}} \log p \sum_{k=2}^{\infty} \frac{1}{p^k} = \sum_{p \leq x^{1/2}} \log p \frac{1}{p(p-1)} \ll \sum_{p \leq x^{1/2}} \frac{\log p}{p^2} \leq \sum_{n=1}^{\infty} \frac{\log n}{n^2}.$$
This last sum converges to a constant, so we must have $\sum_{k=2}^{\infty} \sum_{p \leq x^{1/k}} \frac{\log p}{p^k} = O(1)$. We can thus restrict the von Mangoldt identity to primes, via
$$\sum_{p \leq x} \frac{\log p}{p} = \sum_{n \leq x} \frac{\Lambda(n)}{n} + O(1) = \log x + O(1). \tag{1}$$
Let $\varrho_x := \sum_{p \leq x} (\log p)/p$. Now apply partial summation to σ_x to get
$$\begin{aligned}\sigma_x &= \frac{\varrho_x}{\log x} + O(1) + \int_2^x \frac{\varrho_u}{u(\log u)^2} \mathrm{d}u \\ &= O(1) + \int_2^x \left(\frac{1}{u(\log u)} + O\left(\frac{1}{u(\log u)^2} \right) \right) \mathrm{d}u, \text{ by (1)} \\ &= \log\log x + O(1)\end{aligned}$$
after evaluating the integrals.

2. Prove that
$$\sum_{n \leq x} \omega(n) = x \log\log x + O(x)$$
for $x \geq 3$.

Solution: On expanding the definition of $\omega(n)$ and rearranging sums, we get

$$\begin{aligned}\sum_{n\le x}\omega(n) &= \sum_{n\le x}\sum_{p|n}1\\ &= \sum_{p\le x}\sum_{\substack{n\le x\\ p|n}}1\\ &= \sum_{p\le x}\left\lfloor\frac{x}{p}\right\rfloor\\ &= x\sum_{p\le x}\frac{1}{p}+O\left(\sum_{p\le x}1\right)\\ &= x\log\log x+O(\pi(x)),\end{aligned}$$

where the last line follows from the previous part. The result follows from noting that $\pi(x)=O(x)$ (and is in fact smaller).

3. Let $y=x^{1/2}$ and define

$$\omega'(n)=\sum_{\substack{p|n\\ p\le y}}1.$$

Prove that

$$\omega'(n)\le\omega(n)\le\omega'(n)+1$$

for all integers $n\le x$.

Solution: The function $\omega'(n)$ counts the number of prime factors of n that are less than y. Since this is a subset of all prime factors of n, $\omega'(n)\le\omega(n)$.

It remains to show that $\omega(n)\le\omega'(n)+1$ whenever $n\le x$. Assume by contradiction that $\omega(n)\ge\omega'(n)+2$; then n has at least two prime factors, say p_1 and p_2, which are $>y=x^{1/2}$. But then $n\ge p_1p_2>y^2=x$, which contradicts the assumption that $n\le x$.

4. Prove that

$$\sum_{n\le x}\Big(\omega'(n)-\sum_{p\le y}\frac{1}{p}\Big)^2=x\sigma_y+O(x).$$

(Hint: write

$$\omega'(n)-\sum_{p\le y}\frac{1}{p}=\sum_{p\le y}\Big(\delta_p(n)-\frac{1}{p}\Big),$$

where δ_p is the characteristic function of the integers divisible by p, and then expand the square and handle the various terms separately.)

Solution: Following the hint and expanding the square, we get

$$\begin{aligned}\sum_{n\le x}\Big(\omega'(n)-\sum_{p\le y}\frac{1}{p}\Big)^2 &= \sum_{n\le x}\Big(\sum_{p\le y}\Big(\delta_p(n)-\frac{1}{p}\Big)\Big)^2\\ &= \sum_{n\le x}\sum_{p_1,p_2\le y}(\delta_{p_1}(n)-1/p_1)(\delta_{p_2}(n)-1/p_2)\\ &= \sum_{n\le x}\sum_{p_1,p_2\le y}\Big(\delta_{p_1}(n)\delta_{p_2}(n)-\frac{\delta_{p_1}(n)}{p_2}-\frac{\delta p_2(n)}{p_1}+\frac{1}{p_1p_2}\Big)\end{aligned}$$

We will now split this into four terms which we handle separately. For the second term, we have

$$\begin{aligned}-\sum_{p_1,p_2\le y}\sum_{n\le x}\frac{\delta_{p_1}(n)}{p_2} &= -\sum_{p_1,p_2\le y}\frac{1}{p_2}\left\lfloor\frac{x}{p_1}\right\rfloor\\ &= -x\sum_{p_1,p_2\le y}\frac{1}{p_1p_2}+O\left(\sum_{p_2\le y}\frac{1}{p_2}\right),\end{aligned}$$

using the expansion that $\lfloor z\rfloor = z+O(1)$. By definition of σ_y, this is $-x\sigma_y^2+O(\sigma_y)$. By an identical argument, the third term is also $-x\sigma_y^2+O(\sigma_y)$, and the fourth term is precisely $x\sigma_y^2$.

We now handle the first term. If $p_1\ne p_2$, then the number of integers $n\le x$ with $\delta_{p_1}(n)\delta_{p_2}(n)=1$ is $\left\lfloor\frac{x}{p_1p_2}\right\rfloor$. But if $p_1=p_2$, the number of these integers is $\left\lfloor\frac{x}{p_1}\right\rfloor$. We thus have

$$\begin{aligned}\sum_{n\le x}\sum_{p_1,p_2\le y}\delta_{p_1}(n)\delta_{p_2}(n) &= \sum_{p_1\ne p_2\le y}\left\lfloor\frac{x}{p_1p_2}\right\rfloor+\sum_{p\le y}\left\lfloor\frac{x}{p}\right\rfloor\\ &= \sum_{p_1,p_2\le y}\left\lfloor\frac{x}{p_1p_2}\right\rfloor+\sum_{p\le y}\left(\left\lfloor\frac{x}{p}\right\rfloor-\left\lfloor\frac{x}{p^2}\right\rfloor\right).\end{aligned}$$

Similarly to the previous terms, the sum over p_1 and p_2 is $x\sigma_y^2+O(\pi(y)^2)$, whereas the sum over p is

$$\sum_{p\le y}\left(\left\lfloor\frac{x}{p}\right\rfloor-\left\lfloor\frac{x}{p^2}\right\rfloor\right)=\sum_{p\le y}\left(\frac{x}{p}-\frac{x}{p^2}+O(1)\right)=x\sigma_y+O(x).$$

Combining everything, we then get

$$\begin{aligned}&\sum_{n\le x}\Big(\omega'(n)-\sum_{p\le y}\frac{1}{p}\Big)^2\\ &\quad= \sum_{p_1\ne p_2\le y}\left\lfloor\frac{x}{p_1p_2}\right\rfloor+\sum_{p\le y}\left\lfloor\frac{x}{p}\right\rfloor-\sum_{p_1,p_2\le y}\left\lfloor\frac{x}{p_1}\right\rfloor\frac{1}{p_2}-\sum_{p_1,p_2\le y}\left\lfloor\frac{x}{p_2}\right\rfloor\frac{1}{p_1}+\sum_{p_1,p_2\le y}\frac{x}{p_1p_2}\\ &\quad= x\sigma_y^2+O(\pi(y)^2)+x\sigma_y+O(x)-x\sigma_y^2+O(\sigma_y)-x\sigma_y^2+O(\sigma_y)+x\sigma_y^2\\ &\quad= x\sigma_y+O(\pi(y)^2+x+\sigma_y)\\ &\quad= x\sigma_y+O(x),\end{aligned}$$

where in the last step we note that $\pi(y)^2 \ll y^2 = x$ and $\sigma_y = \log\log y \ll \log\log x \ll x$.

5. Deduce that

$$\sum_{n\le x}(\omega(n) - \log\log x)^2 = x(\log\log x) + O(x\sqrt{\log\log x}),$$

for $x \ge 3$. (This is a theorem of Hardy and Ramanujan.)

Solution: From part 3, we know that $\omega(n) = \omega'(n) + O(1)$, and from part 2 we know that $\sigma_y = \log\log y + O(1) = \log\log x + \log\frac{1}{2} + O(1) = \log\log x + O(1)$. Thus

$$\begin{aligned}\sum_{n\le x}(\omega(n) - \log\log x)^2 &= \sum_{n\le x}\left(\omega'(n) - \sigma_y + O(1)\right)^2 \\ &= \sum_{n\le x}(\omega'(n) - \sigma_y)^2 + O\Big(\sum_{n\le x}|\omega'(n) - \sigma_y|\Big) + O(x) \\ &= x\sigma_y + O\Big(\sum_{n\le x}|\omega'(n) - \sigma_y|\Big) + O(x),\end{aligned}$$

where the last line follows from part 4.

We need to bound the first error term, which we can do by applying Cauchy–Schwarz to get

$$\begin{aligned}\sum_{n\le x}|\omega'(n) - \sigma_y| &\le \Big(\sum_{n\le x}1^2\Big)^{1/2}\Big(\sum_{n\le x}|\omega'(n) - \sigma_y|^2\Big)^{1/2} \\ &= x^{1/2}\Big(\sum_{n\le x}(\omega'(n) - \sigma_y)^2\Big)^{1/2} \\ &= x^{1/2}(x\sigma_y + O(x))^{1/2}, \text{ by part 4} \\ &= x^{1/2}(x\log\log x + O(x))^{1/2}, \text{ by part 2} \\ &\ll x\sqrt{\log\log x}.\end{aligned}$$

Thus we get

$$\sum_{n\le x}(\omega(n) - \log\log x)^2 = x\sigma_y + O(x\sqrt{\log\log x}) = x\log\log x + O(x\sqrt{\log\log x}),$$

as desired.

6. Suppose an integer n has size about 10^{100}, and that $\omega(n) = 12$. Is that something remarkable?

Solution: For $n \sim 10^{100}$, the average size of $\omega(n)$ is $\log\log 10^{100} \approx 5.439$. By part 5, the variance of $\omega(n)$ is the same, so the standard deviation of this distribution is $\sqrt{5.439} \approx 2.332$. Thus the value $\omega(n) = 12$ is about 2.81 standard deviations above average. This is unlikely but not unheard of, occuring for about one in every four hundred integers of this size.

3. For $x \geq 1$, define

$$M(x) = \sum_{n \leq x} \mu(n),$$

where μ is the Möbius function.

1. Show that for $\mathrm{Re}(s) > 1$, we have the equality

$$\frac{1}{\zeta(s)} = s \int_1^{+\infty} M(t) t^{-s-1} dt.$$

Solution: For $\mathrm{Re}(s) > 1$, we have $\zeta(s) = \prod_p \left(1 - \frac{1}{p}\right)^{-1}$, so that

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Applying summation by parts (Lemma 3.2.1 with the sequence $a_n = \mu(n)$ and $f(n) = n^{-s}$), we get

$$\sum_{n \geq 1} \frac{\mu(n)}{n^s} = \int_1^{\infty} \left(\sum_{1 \leq n \leq t} \mu(n) \right) s t^{-s-1} \mathrm{d}t,$$

which is precisely the desired equality.

2. Deduce that, if the estimate

$$M(x) = O(x^{\delta})$$

for $x \geq 2$ is true for a certain $\delta > 0$, then $\zeta(s) \neq 0$ for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > \delta$.
Solution: Assume that $M(x) = O(x^{\delta})$ for a certain $\delta > 0$. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \delta$. Then

$$\begin{aligned} s \int_1^{\infty} M(t) t^{-s-1} \mathrm{d}t &= O\left(|s| \int_1^{\infty} t^{-\mathrm{Re}(s) - 1 + \delta} \mathrm{d}t \right) \\ &= O\left(|s| \left(\left. \frac{t^{-\mathrm{Re}(s) + \delta}}{-\mathrm{Re}(s) + \delta} \right|_1^{\infty} \right) \right) \\ &= O\left(\frac{|s|}{\delta - \mathrm{Re}(s)} \right), \end{aligned}$$

where we have used that $\mathrm{Re}(s) > \delta$ and thus $-\mathrm{Re}(s) + \delta - 1 < -1$. In particular, we know that $s \int_1^{\infty} M(t) t^{-s-1} \mathrm{d}t$ always converges to a finite value whenever $\mathrm{Re}(s) > \delta$, so it is a well-defined analytic function in this region. Thus equality in part (1) must hold not just in the region with $\mathrm{Re}(s) > 1$ but for the entire region $\mathrm{Re}(s) > \delta$, which in turn implies that $\frac{1}{\zeta(s)}$ has no poles in this region, and equivalently $\zeta(s)$ has no zeroes.

3. Similarly, prove that if the estimate

$$\sum_{n\leq x} \Lambda(n) = x + O(x^\delta)$$

is valid for some $\delta > 0$, then $\zeta(s) \neq 0$ for all $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > \delta$.
Solution: Again by summation by parts, we have that

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = s\int_1^\infty \Big(\sum_{n\leq t}\Lambda(n)\Big)t^{-s-1}\mathrm{d}t.$$

Applying the assumption and our work from the previous part, we get that for $\mathrm{Re}(s) > \delta$,

$$\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= s\int_1^\infty \Big(\sum_{n\leq t}\Lambda(n)\Big)t^{-s-1}\mathrm{d}t \\
&= s\int_1^\infty t^{-s}\mathrm{d}t + O\left(\frac{|s|}{\delta - \mathrm{Re}(s)}\right) \\
&= \frac{s}{s-1} + O\left(\frac{|s|}{\delta - \mathrm{Re}(s)}\right).
\end{aligned}$$

In the region where $\mathrm{Re}(s) > \delta$, we therefore get that $-\frac{\zeta'(s)}{\zeta(s)}$ has a unique pole at $s = 1$, which is simple, coming from the pole of $\zeta(s)$ at $s = 1$. Any zero of $\zeta(s)$ in this region would induce another pole, so $\zeta(s)$ cannot have any zeroes in this region.

4. The ternary divisor function d_3 is defined as the triple Dirichlet convolution $1 \star 1 \star 1$.

1. Compute the Dirichlet generating series $D(s)$ for d_3 and prove that it has meromorphic continuation to $\mathrm{Re}(s) > 0$ with a triple pole at $s = 1$.
Solution: The identity function 1 has polynomial growth, and thus so does $1\star 1 = d$ and $d_3 = 1\star 1\star 1 = 1\star d$, so the Dirichlet generating series $D(s)$ is given by

$$D(s) = \left(\sum_{n=1}^{\infty}\frac{1}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n^s}\right) = \zeta(s)^3.$$

The meromorphic continuation of $D(s)$ to $\mathrm{Re}(s) > 0$ is thus the cube of the meromorphic continuation of $\zeta(s)$ to the same region, so it exists, and has a unique pole of order 3 at $s = 1$ coming from the cube of the pole of $\zeta(s)$ at $s = 1$.

2. Let $\epsilon > 0$ be a real number. Prove that $d_3(n) \ll n^\epsilon$ for $n \geq 1$.
Solution: This follows along the lines of exercise 2.1 from exercise sheet 5, and just like for that problem, multiple proofs are available.
Note that d_3 is multiplicative, and for p prime and $k \geq 1$, $v \geq 2$ integers, we have

$$d_3(p)^k = \Big(\sum_{\substack{a_1,a_2,a_3\in\mathbb{N} \\ a_1a_2a_3=p}} 1\Big)^k = 3^k$$

and

$$d_3(p^v)^k = \binom{v+2}{3}^k \ll v^{3k}.$$

Thus we can apply Proposition 3.6.2 with $A = 3k$ to show that

$$\sum_{n=1}^{\infty} \frac{d_3(n)^k}{n^s} = \zeta(s)^{3^k} D^{\sharp}_{d_3^k}(s)$$

for $\mathrm{Re}(s) > 1/2$, where $D^{\sharp}_{d_3^k}(s)$ is holomorphic for $\mathrm{Re}(s) > 1/2$. Thus in particular, $\sum_{n=1}^{\infty} \frac{d_3(n)^k}{n^s}$ converges for $\mathrm{Re}(s) > 1$ for all $k \geq 1$, so by exercise 2.1 from exercise sheet 5, $d_3(n) \ll_\epsilon n^\epsilon$ for all $\epsilon > 0$.

3. Let δ be a real number with $0 < \delta < 1$. Prove that

$$D_f(s) \ll (1 + |s|)^3$$

for $\mathrm{Re}(s) \geq \delta$ and $|\mathrm{Im}(s)| \geq 1$.

<u>Solution:</u> For all $s = \sigma + it$ with $\mathrm{Re}(s) = \sigma \geq \delta > 0$ and $|\mathrm{Im}(s)| = |t| \geq 1$, we have

$$\begin{aligned} |D_f(s)| &= |\zeta(s)|^3 \\ &\leq \left(\frac{|s|}{|s-1|} + \frac{|s|}{\sigma}\right)^3, \text{ by Prop 3.6.2 (1)} \\ &\ll \left(\frac{\sqrt{\sigma^2+t^2}}{\sqrt{(\sigma-1)^2+t^2}} + |s|\frac{1}{\delta}\right)^3. \end{aligned}$$

The first fraction approaches 1 when σ or t grows large, and is maximized when $\sigma = 1$ and $|t| = 1$ (recalling that $|t| \geq 1$). In this case $\sqrt{\sigma^2+t^2}/\sqrt{(\sigma-1)^2+t^2} = 2$; in particular, the fraction is $O(1)$ in the region under consideration. Thus

$$|D_f(s)| \ll \left(1 + |s|\frac{1}{\delta}\right)^3 \ll_\delta (1+|s|)^3.$$

4. Let $\epsilon > 0$ be a real number. Using Mellin transform methods, prove that

$$\sum_{n \leq x} d_3(n) = x f(\log x) + O(x^{4/5+\epsilon})$$

for $x \geq 2$, where f is a polynomial of degree 2 with leading term $X^2/2$.

<u>Solution:</u> We will follow the procedure from the proof of Proposition 3.6.4. Many details are identical, so in some places we will be brief.

For a parameter λ with $0 < \lambda < x$, let $\varphi : [0, +\infty[\to [0,1]$ be a smooth function such that $\varphi(t) = 0$ for $t \geq x + \lambda$, such that $\varphi(t) = 1$ for $0 \leq t \leq x$, and such that for every integer $j \geq 0$, $\varphi^{(j)}(t) = O(\lambda^{-j})$, with the implied constant depending only on j.

Let $\hat{\varphi}(s)$ denote the Mellin transform of $\varphi(t)$, which satisfies the fast-decay bound

$$\hat{\varphi}(s) \ll x^{\sigma} \left(\frac{x}{\lambda}\right)^{m-1} (1+|t|)^{-m} \tag{2}$$

for any integer $m \geq 1$.

For $\mathrm{Re}(s) > 1$, the Dirichlet series $D_f(s)$ converges absolutely, so

$$\sum_{n\geq 1} d_3(n)\varphi(n) = \frac{1}{2\pi i}\int_{(2)} D_f(s)\hat{\varphi}(s)\mathrm{d}s,$$

since the integrand is integrable by the fast decay of the Mellin transform of φ. Now let $0 < \delta < 1/2$ be a fixed positive real number; by part 3, $|D_f(s)| \ll_{\delta} (1+|s|)^3$ whenever $\mathrm{Re}(s) \geq \delta$ and $|\,\mathrm{Im}(s)| \geq 1$.

We now apply Cauchy's theorem to the rectangle with vertices $2-iT, 2+iT, \delta+iT, \delta-iT$, oriented counterclockwise, for a parameter $T \geq 1$. The result is

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT} D_f(s)\hat{\varphi}(s)\mathrm{d}s + \frac{1}{2\pi i}\int_{2+iT}^{\delta+iT} D_f(s)\hat{\varphi}(s)\mathrm{d}s$$
$$\frac{1}{2\pi i}\int_{\delta+iT}^{\delta-iT} D_f(s)\hat{\varphi}(s)\mathrm{d}s + \frac{1}{2\pi i}\int_{\delta-iT}^{2-iT} D_f(s)\hat{\varphi}(s)\mathrm{d}s = \mathrm{res}_{s=1} D_f(s)\hat{\varphi}(s). \tag{3}$$

For $\delta \leq \sigma \leq 2$, we have

$$\hat{\varphi}(\sigma+iT) \ll (1+T)^{-4}, \hat{\varphi}(\sigma-iT) \ll (1+T)^{-4},$$
$$D_f(\sigma+iT) \ll (1+T)^3, D_f(\sigma-iT) \ll (1+T)^3,$$

so

$$\frac{1}{2\pi i}\int_{2+iT}^{\delta+iT} D_f(s)\hat{\varphi}(s)\mathrm{d}s + \frac{1}{2\pi i}\int_{\delta-iT}^{2-iT} D_f(s)\hat{\varphi}(s)\mathrm{d}s \ll T^{-1},$$

and thus taking $T \to +\infty$ we obtain that we can shift the contour from the vertical line with real part 2 to the vertical line with real part δ, obtaining from (3) that

$$\frac{1}{2\pi i}\int_{(2)} D_f(s)\hat{\varphi}(s)\mathrm{d}s = \mathrm{res}_{s=1} D_f(s)\hat{\varphi}(s) + \frac{1}{2\pi i}\int_{(\delta)} D_f(s)\hat{\varphi}(s)\mathrm{d}s.$$

We now estimate the shifted integral. Applying (2) with $m = 5$, we get that

$$\frac{1}{2\pi i}\int_{(2)} D_f(s)\hat{\varphi}(s)\mathrm{d}s \ll x^{\delta}\left(\frac{x}{\lambda}\right)^4 \int_{\mathbb{R}} (1+|t|)^{-2}\mathrm{d}t \ll x^{\delta}\left(\frac{x}{\lambda}\right)^4.$$

Next we compute the residue. In a neighborhood of $s = 1$ we have, for suitable constants c_0 and c_1, the Laurent expansion

$$\zeta(s) = \frac{1}{s-1} + c_0 + c_1(s-1) + O((s-1)^2),$$

and thus

$$D_f(s) = \zeta(s)^3 = \frac{1}{(s-1)^3} + \frac{3c_0}{(s-1)^2} + \frac{(3c_0^2+3c_1)}{(s-1)} + O(1),$$

whereas

$$\hat{\varphi}(s) = \hat{\varphi}(1) + (s-1)\hat{\varphi}'(1) + \frac{(s-1)^2}{2}\hat{\varphi}''(1) + O((s-1)^3).$$

We thus have, recalling that $\mathrm{res}_{s=1}$ is the coefficient of the $\frac{1}{s-1}$ term in the Laurent expansion around $s = 1$, that

$$\begin{aligned}\mathrm{res}_{s=1} D_f(s)\hat{\varphi}(s) &= \mathrm{res}_{s=1}\zeta(s)^3\hat{\varphi}(s) \\ &= \frac{\hat{\varphi}''(1)}{2} + 3c_0\hat{\varphi}'(1) + (3c_0^2 + 3c_1)\hat{\varphi}(1).\end{aligned}$$

In order to compute the residue, it remains to compute the derivatives of $\hat{\varphi}$ at 1, which can be done explicitly. We have, as in the notes,

$$\hat{\varphi}(1) = \int_{\mathbb{R}} \varphi(y)\mathrm{d}y = x + O(\lambda), \quad \hat{\varphi}'(1) = \int_{\mathbb{R}} \varphi(y)\log y\mathrm{d}y = x\log x - x + O(\lambda\log x).$$

Continuing to the second derivative, we have

$$\begin{aligned}\hat{\varphi}''(1) = \int_{\mathbb{R}} \varphi(y)(\log y)^2\mathrm{d}y &= \int_0^x (\log y)^2\mathrm{d}y + O(\lambda(\log x)^2) \\ &= (y(\log y)^2|_0^x - \int_0^x (2\log y)\mathrm{d}y + O(\lambda(\log x)^2) \\ &= x(\log x)^2 - 2x\log x + 2x + O(\lambda(\log x)^2).\end{aligned}$$

Combining all terms, the residue is given by

$$\mathrm{res}_{s=1} D_f(s)\hat{\varphi}(s) = xg(\log x) + O(\lambda(\log x)^2),$$

where $g(y)$ is the quadratic polynomial given by

$$g(y) = \frac{y^2}{2} + (3c_0 - 1)y + 3c_0^2 + 3c_1 - 3c_0 + 1.$$

We have now obtained the formula

$$\sum_{n\geq 1} d_3(n)\varphi(n) = xg(\log x) + O\left(\lambda(\log x)^2 + x^\delta\left(\frac{x}{\lambda}\right)^4\right).$$

Choosing $\lambda = x^{4/5}$ to get that the error term is $O(x^{4/5+\delta})$. Finally we then have

$$\sum_{n\leq x} d_3(n) = \sum_{n\geq 1} d_3(n)\varphi(n) + O(\lambda x^\delta) = xg(\log x) + O(x^{4/5+\delta})$$

for any $\delta > 0$, where the implied constant depends on δ and where g is a quadratic polynomial with leading term 1/2.

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