Useful facts about Lie Groups

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We will see early on in the course that Symmetric Spaces are quotients of certain Lie groups — it shouldn't be too surprising therefore that a solid understanding of Lie groups will be useful. If you've taken a course about Lie Groups your knowledge will be more than sufficient; if you haven't, these notes are meant to fill the gaps. They can be skimmed through and then referred back to when needed throughout the course, a passing familiarity with the definitions is already very good for the course!

As such we will make some simplifications and skip a lot of details (including all the proofs). Prof. A. Iozzi's Lie Groups notes (available on the course website) can be consulted when needed — the statements should match very closely. You are encouraged to attempt all the exercises[1]. Also contact if there are any mistakes or if anything isn't clear; these notes are very brief and so there are a few concepts and definitions I miss out.

1 Lie groups and Lie algebras

A Lie group is a group G that has a compatible manifold structure. That is, a Lie group is a manifold G with smooth multiplication and inversion maps.
Since it is a smooth manifold we can do Differential Geometry on it, I'll refer to certain important concepts in passing. We will need some of these for the course, but they will introduced properly in the lectures. I will mention some in passing below, as ever please don't hesitate to contact if there are lingering doubts.

First examples of Lie groups:

(a) A countable[2] discrete group is a 0-dimensional Lie group.

(b) $(\mathbb{R}, +)$ is a Lie group, as is $(\mathbb{R}^n, +)$.

(c) $(\mathbb{R}^*, \times)$ is a Lie group, as is $((\mathbb{R}^*)^n, \times)$ with pointwise multiplication.

(d) $\mathrm{GL}(n, \mathbb{R})$ is a Lie group. It is an open subset of $\mathbb{R}^{n\times n}$ (given by the *nonvanishing* of det) and so has dimension n^2. More generally for a finite dimensional $\mathbb{R}$-vector space V we have the Lie group $\mathrm{GL}(V)$.

(e) $\mathrm{SL}(n, \mathbb{R})$ is a Lie group. Using the Inverse Function Theorem we see that it has dimension $n^2 - 1$.

[1] I don't plan on writing solutions, but please feel free to contact if you would like any, or want any hints.
[2] By convention our manifolds are second countable.

(f) $\mathrm{O}(n,\mathbb{R})$ is a Lie group of dimension $\frac{n(n-1)}{2}$.

(g) $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is a Lie group.

(h) A product of Lie groups is again a Lie group (so for example, $\mathbb{T}^n : (S^1)^n$ is a compact Lie group).

Exercise 1. Verify the dimensions of $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{O}(n,\mathbb{R})$.

For any $g \in G$, consider the map $L_g : G \to G : x \mapsto gx$, which we call *left translation by g*. This is a diffeomorphism of G, and so in particular gives an isomorphism $d_eL_g : T_eG \to T_gG$. That is, locally G looks everywhere the same, and we might hope that T_eG contains some interesting algebraic information connected to the group. In fact we can transfer any vector in T_eG to one in T_gG — and so we can construct a G-invariant vector field. This gives a natural identification between T_eG and $\mathrm{Vect}(G)^G$, the space of G-invariant vector fields on G.
We hope that T_eG retains some of the algebraic structure of the group G, at least locally. One way to obtain an algebraic structure on T_eG is to transport the Lie bracket from $\mathrm{Vect}(G)^G$ via the above identification, and indeed this is what we do in general. For our purposes, the concrete example below will suffice.

Example 1. Consider the tangent space at the identity of $\mathrm{GL}(n,\mathbb{R})$, it is just the space of $n\times n$ matrices $M_n(\mathbb{R})$, which for reasons that will become clear in a moment we denote by $\mathfrak{gl}(n,\mathbb{R})$. We have a natural map $\exp : \mathfrak{gl}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R}) : X \mapsto I + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \cdots$, the *matrix exponential*. It is classical that this converges for all matrices, and for every matrix has an invertible image.
Similarly in a small neighbourhood U of the identity we have the logarithm map

$$\log : U \to \mathfrak{gl}(n,\mathbb{R}) : (I + A) \mapsto A - \tfrac{1}{2}A^2 + \tfrac{1}{3}A^3 + \cdots$$

We might hope that $\exp(X)\exp(Y) = \exp(X+Y)$, but there is no reason for this to hold (unless for some reason X and Y commute). But we can try (at least formally, or for X, Y very close to 0) to evaluate the left hand side:

$$\begin{aligned}\exp(X)\exp(Y) &= \Big(I + X + \frac{X^2}{2} + \cdots\Big)\Big(I + Y + \frac{Y^2}{2} + \cdots\Big)\\ &= 1 + (X+Y) + XY + \frac{X^2}{2} + \frac{Y^2}{2} + \text{higher order terms}\end{aligned}$$

and therefore

$$\begin{aligned}\log(\exp(X)\exp(Y)) &= (X+Y) + XY + \frac{X^2}{2} + \frac{Y^2}{2} - \frac{1}{2}(X+Y)^2 + \text{higher order terms}\\ &= (X+Y) + \frac{XY - YX}{2} + \text{higher order terms}\end{aligned}$$

and so the quantity $[X,Y] := XY - YX$ helps us multiply $\exp(X)$ and $\exp(Y)$ — this quantity 'remembers' the algebraic structure of $\mathrm{GL}(n,\mathbb{R})$, at least close to the identity.

Remark 2. In fact we can write $\exp(X)\exp(Y) = \exp(Z)$ where $Z = X + Y + [X,Y] + \cdots$ where all the higher order terms involve recursions of $[\cdot,\cdot]$. This is known as the *Baker-Campbell-Hausdorff formula*.

Remark 3. Note that $\exp((t_1+t_2)X) = \exp(t_1X)\exp(t_2X)$ and $\exp(tX)^{-1} = \exp(-tX)$ and so $\exp : \mathbb{R} \to \mathrm{GL}(n,\mathbb{R})$ is a (smooth) homomorphism, called a *one-parameter subgroup* of $\mathrm{GL}(n,\mathbb{R})$.

Definition 4. A *Lie algebra* is a finite dimensional vector space $\mathfrak{g}$ over $\mathbb{R}$ with a bilinear operation $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ such that

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$$

A morphism of Lie algebras is a linear map between them that commutes with the bracket operation.

Exercise 2. Verify that $\mathfrak{gl}(n,\mathbb{R})$ with $[X,Y] \coloneqq XY - YX$ is a Lie algebra.

The relation given in the definition is known as the *Jacobi identity*.

Exercise 3. Prove using the Jacobi identity that $[\cdot,\cdot]$ must be antisymmetric.

Remark 5. The Jacobi identity might seem a bit mysterious, but we have the following intuition: Consider the 'differential' $\delta_X : \mathfrak{g} \to \mathfrak{g} : Y \mapsto [X,Y]$. With respect to this, the Jacobi identity is just the Leibniz rule (replacing products with Lie brackets).

Similarly to the example of $\mathrm{GL}(n,\mathbb{R})$ we can associate to every Lie group G a bracket on T_eG turning it into a Lie algebra, which we will denote $\mathfrak{g}$. This is a natural construction, as the followng Theorem shows.

Theorem 6. *Let $\varphi : H \to G$ be a Lie group homomorphism (that is, a group homomorphism that is smooth). Then $d_e\varphi : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra morphism.*

Remark 7. This theorem allows us to compute the Lie bracket for Lie groups such as $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n,\mathbb{R})$, (at least once we know their tangent space at the identity — there is an exercise about this at the end), since they are Lie subgroups of $\mathrm{GL}(n,\mathbb{R})$ (use the inclusion map).

Recall that we introduced the Lie bracket in the hope that it preserves some algebraic structure from G at least close to the identity — and indeed this is true:

Theorem 8. *Suppose two Lie groups G, G' have isomorphic Lie algebras, then they are locally isomorphic. That is, there are neighbourhoods of the identity $e \in U \subset G$, $e \in V \subset G'$ and a diffeomorphism $\varphi : U \to V$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ whenever $x, y, xy \in U$.*

In fact we can ensure that the derivative of this local isomorphism at e is the given Lie algebra isomorphism. We also make note of the following fact:

Theorem 9 (Ado)**.** *Any Lie algebra is a Lie subalgebra of some $\mathfrak{gl}(n,\mathbb{R})$.*

Where a Lie subalgebra is a vector subspace $\mathfrak{h} \leq \mathfrak{g}$ such that $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$.
There is a clear relationship between a homomorphism $\varphi : G \to H$ and its derivative $d_e\varphi : \mathfrak{g} \to \mathfrak{h}$. It turns out that the exponential[3] is a natural transformation between the two.

Proposition 10 (Naturality of exp)**.** *Let $\varphi : H \to G$ be a Lie group homomorphism. Then the following diagram commutes:*

[3] We only defined it for $\mathrm{GL}(n,\mathbb{R})$! But not to worry — there is a general way to define it, and it coincides with our definition for closed subgroups of $\mathrm{GL}(n,\mathbb{R})$

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & G \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{h} & \xrightarrow{d_e\varphi} & \mathfrak{g} \end{array}$$

Remark 11. It might be tempting to conclude from Ado's Theorem (Theorem 9) that all Lie Groups are (isomorphic to) closed subgroups of some $\mathrm{GL}(n, \mathbb{R})$. This is not the case, for example the universal cover of $\mathrm{SL}(2, \mathbb{R})$ is a Lie group that doesn't embed in any $\mathrm{GL}(n, \mathbb{R})$. However from Ado's Theorem and Theorem 8 we can conclude that every Lie group is locally isomorphic to a Lie subgroup of $\mathrm{GL}(m, \mathbb{R})$.

We note however all the groups we will be considering in this course will in fact be closed subgroups of some $\mathrm{GL}(n, \mathbb{R})$. This is a fact about *semisimple* groups, which we will come back to later. So for our purposes, thinking of everything as matrices is sufficient.

We list some useful facts in the following theorem:

Theorem 12. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$.*

(a) If H is a closed subgroup of G then it is also a Lie group;

(b) If G is connected and abelian then it is isomorphic to $\mathbb{T}^n \times \mathbb{R}^m$. It is simply connected if and only if $n = 0$, and compact if and only if $m = 0$;

(c) If $\mathfrak{h}' \leq \mathfrak{g}$ is a Lie subalgebra, then there is a unique pair (H, φ) where H is a Lie group and $\varphi : H \to G$ is an injective group homomorphism such that $\varphi(H)$ is an immersed submanifold, such that $d\varphi(\mathfrak{h}) = \mathfrak{h}'$;

(d) φ as above is an embedding if and only if $\varphi(H)$ is closed in G;

(e) A closed subgroup $H \leq G$ is is normal if and only if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

Definition 13. A subspace $\mathfrak{h} \leq \mathfrak{g}$ is an *ideal* if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$.

2 All things adjoint

You will have seen (or will soon see, I'm sure), that understanding the representations of a group G goes a long way to understanding the group itself. By a representation (in this course) we mean a smooth homomorphism $\rho : G \to \mathrm{GL}(V)$, where V is for some finite dimensional vector space. Given such a representation, we get an associated representation of the Lie algebra, given by $d_e\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. That is,

$$d_e(\rho)(X)(v) := \frac{d}{dt}\Big|_{t=0} \rho(\exp(tX))(v)$$

Remark 14. Checking that such a map is smooth might seem daunting, but it is a fact that any continuous (in fact, any measurable) homomorphism between Lie groups is automatically smooth.

Exercise 4. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of connected Lie group G, and $W \leq V$ a subspace. Show that W is $\rho(G)$-invariant if and only if it is $d_e\rho(\mathfrak{g})$-invariant.
Hint: You should use the naturality of exp*, and the fact that any element of $g \in G$ can be written as the product of* $\exp(X_1) \cdots \exp(X_k)$ *for some $X_i \in \mathfrak{g}$.*

There is an important representation we will look at, which comes from a natural group action. For any $g \in G$, consider $c_g : G \to G : h \mapsto ghg^{-1}$.

Exercise 5. Show that $d_e c_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie Algebra automorphism.

Definition 15. Let G be a Lie group.

(a) The *Adjoint representation* of G is $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}) : g \mapsto d_e c_g$;

(b) The *adjoint representation* of $\mathfrak{g}$ is $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ defined by $\mathrm{ad} = d_e \mathrm{Ad}$.

This all seems confusing, but in fact we can say what these representations are concretely (at least in many cases of interest).

Proposition 16. *(a) If $G \leq \mathrm{GL}(n, \mathbb{R})$ is a closed subgroup then for $g \in G$ and $X \in \mathfrak{g}$,*

$$\mathrm{Ad}(g)(X) = gXg^{-1}$$

(b) If G is any Lie group, and $X, Y \in \mathfrak{g}$, then

$$\mathrm{ad}(X)(Y) = [X, Y]$$

In fact, this is often given as the definition of the adjoint representation (and you can think of it as such).

Consider the centre $Z(G)$ of a closed subgroup $G \leq \mathrm{GL}(n, \mathbb{R})$. It is clear that if $g \in Z(G)$ then $g \in \ker \mathrm{Ad}$. In fact the converse is also true, for any Lie group we have

$$Z(G) = \ker \mathrm{Ad} \quad \text{and} \quad Z(\mathfrak{g}) = \ker \mathrm{ad}$$

3 Semisimplicity

Definition 17. A Lie algebra is *simple* if it isn't abelian and its only ideals are $\{0\}$ and $\mathfrak{g}$. It is *semisimple* if it is the direct sum of simple ideals.

We say a connected group G is *(semi)simple* if $\mathfrak{g}$ is.

Proposition 18. *(a) G is a connected simple Lie group if and only if every connected normal proper subgroup is trivial.*

(b) G is a connected semisimple Lie group if and only if every connected normal abelian subgroup is trivial.

Exercise 6. Show that if $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

There is an important bilinear form associated with $\mathfrak{g}$.

Definition 19. The *Killing form* is $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} : (X, Y) \mapsto \mathrm{Tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$.

Exercise 7. Show that $B_{\mathfrak{g}}$ is independent of the choice of basis on $\mathfrak{g}$, and that $B_{\mathfrak{g}}(X, Y) = B_{\mathfrak{g}}(Y, X)$.

Exercise 8. Show that $B_{\mathfrak{g}}$ is ad-invariant[4]. That is, for all $X, Y, Z \in \mathfrak{g}$,

$$B_{\mathfrak{g}}(\mathrm{ad}(X)(Y), Z) + B_{\mathfrak{g}}(Y, \mathrm{ad}(X)(Z)) = 0$$

The key use for us is that the Killing form can detect whether a Lie algebra is semisimple.

Theorem 20 (Dieudonné)**.** *$\mathfrak{g}$ is semisimple if and only if $B_{\mathfrak{g}}$ is non-degenerate.*

Notice that non-degenerate doesn't mean definite. This only happens in a specific case:

Theorem 21. *Let G be a connected semisimple Lie group. Then the following are equivalent:*

(a) G is compact;

(b) $B_{\mathfrak{g}}$ is negative definite;

(c) $B_{\mathfrak{g}}$ is definite.

4 Further exercises

Exercise 9. Show that the derivative of the determinant function is the trace function.

Exercise 10. Compute the Lie algebras of the following groups:

(a) $\mathrm{SL}(n, \mathbb{R})$;

(b) $\mathrm{O}(n, \mathbb{R})$;

(c) $\mathrm{O}(p, q)$ (the real matrices that preserve a quadratic form of signature (p, q)).

For (b) and (c) you may use the following fact:
If $A, B : (-\varepsilon, \varepsilon) \to \mathbb{R}^{n\times n}$ are smooth curves and $\varphi(s) := A(s)B(s)$, then

$$\varphi'(s) = A'(s)B(s) + A(s)B'(s)$$

Exercise 11. Let $\mathfrak{g} = \bigoplus_{i\in I} \mathfrak{g}_i$ be the direct sum of simple ideals. Show that any ideal $\mathfrak{h} \trianglelefteq \mathfrak{g}$ is of the form $\mathfrak{h} = \bigoplus_{j\in J} \mathfrak{g}_j$ with $J \subset I$.

Remark 22. This implies immediately:

(i) Any semisimple Lie algebra has a finite number of ideals.

(ii) Any connected semisimple Lie group with finite center has a finite number of connected normal subgroups.

[4]This might seem like an odd way to define 'invariance', but it is simply the derivative of Ad-invariance.