Exercise Sheet 0

This exercise sheet is meant to remind you of the basics about Lie groups — it goes with the notes 'Useful facts about Lie groups', and all the necessary definitions are there. You are encouraged to attempt all the exercises, and contact Segev if you want to discuss anything.

Exercise 1. Verify that $\dim(\mathrm{SL}(n,\mathbb{R})) = n^2 - 1$ and $\dim(\mathrm{O}(n,\mathbb{R})) = \frac{n(n-1)}{2}$.

Exercise 2. Verify that $\mathfrak{gl}(n,\mathbb{R})$ with $[X,Y] := XY - YX$ is a Lie algebra.

Exercise 3. Prove using the Jacobi identity that $[\cdot,\cdot]$ must be antisymmetric.

Exercise 4. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of connected Lie group G, and $W \leq V$ a subspace. Show that W is $\rho(G)$-invariant if and only if it is $d_e\rho(\mathfrak{g})$-invariant.
Hint: You should use the naturality of $\exp$*, and the fact that any element of* $g \in G$ *can be written as the product of* $\exp(X_1)\cdots\exp(X_k)$ *for some* $X_i \in \mathfrak{g}$.

Exercise 5. Show that $d_e c_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie Algebra automorphism.

Exercise 6. Show that if $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$.

Exercise 7. Show that $B_\mathfrak{g}$ is independent of the choice of basis on $\mathfrak{g}$, and that $B_\mathfrak{g}(X,Y) = B_\mathfrak{g}(Y,X)$.

Exercise 8. Show that $B_\mathfrak{g}$ is ad-invariant. That is, for all $X,Y,Z \in \mathfrak{g}$,

$$B_\mathfrak{g}(\mathrm{ad}(X)(Y),Z) + B_\mathfrak{g}(Y,\mathrm{ad}(X)(Z)) = 0$$

Exercise 9. Show that the derivative of the determinant function is the trace function.

Exercise 10. Compute the Lie algebras of the following groups:

(a) $\mathrm{SL}(n,\mathbb{R})$;

(b) $\mathrm{O}(n,\mathbb{R})$;

(c) $\mathrm{O}(p,q)$ (the real matrices that preserve a quadratic form of signature (p,q)).

For (b) and (c) you may use the following fact:
If $A,B : (-\varepsilon,\varepsilon) \to \mathbb{R}^{n\times n}$ *are smooth curves and* $\varphi(s) := A(s)B(s)$*, then*

$$\varphi'(s) = A'(s)B(s) + A(s)B'(s)$$

Exercise 11. Let $\mathfrak{g} = \bigoplus_{i\in I}\mathfrak{g}_i$ be the direct sum of simple ideals. Show that any ideal $\mathfrak{h} \trianglelefteq \mathfrak{g}$ is of the form $\mathfrak{h} = \bigoplus_{j\in J}\mathfrak{g}_j$ with $J \subset I$.

Remark. This implies immediately:

(i) Any semisimple Lie algebra has a finite number of ideals.

(ii) Any connected semisimple Lie group with finite center has a finite number of connected normal subgroups.