Solutions Exercise Sheet 1

1. Let $G = \mathrm{SL}_2(\mathbb{R})$. The aim of this exercise is to show that $G_\mathbb{Z} = \mathrm{SL}_2(\mathbb{Z})$ is a lattice in G.

   (a) Argue that $G_\mathbb{Z}$ is discrete in G and that both G and $G_\mathbb{Z}$ are unimodular.

Solution. $\mathrm{SL}(n,\mathbb{Z})$ is contained in $\mathbb{Z}^{n\times n}$, which is a discrete subset of $\mathbb{R}^{n\times n}$; it follows that $G_\mathbb{Z}$ is discrete in G by restriction from $\mathbb{R}^{n\times n}$ to G. In addition, $G_\mathbb{Z}$ is unimodular since it is discrete (and so the Haar measure is the counting measure). For G we recall the fact that $|\det(x_{ij})|^{-1}\, dx_{11}\cdots dx_{nn}$ is the bi-invariant Haar measure in $\mathrm{GL}(n,\mathbb{R})$ and that G is a closed normal subgroup of $\mathrm{GL}(n,\mathbb{R})$ (see the Lie Groups lecture notes, Proposition 2.3, on page 28).

From this we know that $G/G_\mathbb{Z}$ admits a nonzero G-invariant measure μ which is unique up to a non-zero constant. In order to show that $G_\mathbb{Z}$ is a lattice we have to show that $\mu(G/G_\mathbb{Z}) < \infty$. For this, we will use the following fact:

   (b) Assume that there exists a measurable set $A \subseteq G$ of finite measure such that every $G_\mathbb{Z}$-orbit intersects A (that is, for every $g \in G$ there exists some $\gamma \in G_\mathbb{Z}$ such that $g\gamma \in A$). Show that $\mu(G/G_\mathbb{Z})$ is finite.

Solution. Recall Weil's formula (Lie Groups notes, Theorem 2.4 on page 34) which for the (integrable) characteristic function χ_A of A in G states that

$$\mu(A) = \int_G \chi_A(g)\, dg = \int_{G/G_\mathbb{Z}} \left(\int_{G_\mathbb{Z}} \chi_A(\gamma h)\, dh \right) d(\gamma G_\mathbb{Z})$$

By the assumption the inner integral is always greater than some absolute constant depending only on the Haar measure of $G_\mathbb{Z}$, and so we infer that $\mu(A) \geq c\mu(G/G_\mathbb{Z})$.

   (c) Show that the map sending

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \text{ to } z \longmapsto g\cdot z := \frac{az+b}{cz+d}$$

   is a group homomorphism $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{Bih}(\mathbb{H}^2)$, where $\mathrm{Bih}(\mathbb{H}^2)$ denotes the biholomorphic maps of the complex upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} |\, \mathrm{Im}(z) > 0\}$. Show that its kernel is $\{\pm I\}$ where I denotes as usual the 2×2 identity matrix.

   These maps are known as *Möbius transformations.*

Solution. Define the automorphy factor $j : \mathrm{SL}(2,\mathbb{R}) \times \mathbb{H}^2 \to \mathbb{C}$ by

$$j(\gamma, z) = (cz + d) \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

For all $z \in \mathbb{H}$ one has the trivial matrix relations

$$\gamma \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az + b \\ cz + d \end{pmatrix} = j(\gamma, z) \begin{pmatrix} \gamma z \\ 1 \end{pmatrix}$$

Given $\alpha, \beta \in \mathrm{SL}(2,\mathbb{R})$ one now computes $\alpha\beta \begin{pmatrix} z \\ 1 \end{pmatrix}$ in two different ways: this yields both

$$\alpha\beta \begin{pmatrix} z \\ 1 \end{pmatrix} = j(\alpha\beta, z) \begin{pmatrix} (\alpha\beta)z \\ 1 \end{pmatrix} \quad \& \quad \alpha\beta \begin{pmatrix} z \\ 1 \end{pmatrix} = j(\alpha, \beta z) j(\beta, z) \begin{pmatrix} \alpha(\beta z) \\ 1 \end{pmatrix}$$

It follows that we have the automorphy relation $j(\alpha\beta, z) = j(\alpha, \beta z) j(\beta, z)$ (also commonly referred to as the *cocycle condition*) and furthermore that $(\alpha\beta)z = \alpha(\beta z)$. Hence $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{Bih}(\mathbb{H}^2)$ is indeed a homomorphism, the remaining assertions are easy to verify.

(d) Prove that the induced homomorphism

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) / \{\pm I\} \to \mathrm{Bih}(\mathbb{H}^2)$$

of (c) is actually an isomorphism. For the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}^2$ from above determine the orbit Gi and stabilizer K of $i \in \mathbb{H}^2$. (Show also that K is compact.) Using this, show that we have a diffeomorphism

$$G/K \longrightarrow \mathbb{H}^2, \; g \longmapsto g \cdot i.$$

Solution. It suffices to show that any biholomorphism of $\mathbb{H}^2$ is actually a Möbius map. Note that the Cayley transform $\varphi : z \mapsto \frac{z-i}{z+i}$ maps the upper-half plane $\mathbb{H}^2$ biholomorphically to the (open) unit disc $\mathbb{D}$, and so we have an isomoprhism $\mathrm{Bih}(\mathbb{H}^2) \to \mathrm{Bih}(\mathbb{D}^2) : \psi \mapsto \varphi \circ \psi \circ \varphi^{-1}$.
As such, we identify the $\mathrm{SL}(2,\mathbb{R})$ action on $\mathbb{H}^2$ with the $\mathrm{SU}_{1,1}(\mathbb{C})$ action on $\mathbb{D}$, and so it suffices to show that the latter is the group of biholomorphisms of $\mathbb{D}$. The $\mathrm{SU}_{1,1}(\mathbb{C})$-action is given by

$$\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \cdot z = \frac{az + b}{\overline{b}z + \overline{a}} \qquad (\text{where } |a|^2 - |b|^2 = 1)$$

(You should verify this carefully!)

Now let φ be an arbitrary biholomorphism of $\mathbb{D}$, let $b = -\varphi(0)$ and set

$$\psi = \frac{1}{1+|b|^2}\begin{pmatrix}1 & b\\ \bar{b} & 1\end{pmatrix}\circ\varphi$$

This is a biholomorphism of $\mathbb{D}$ which fixes 0. The classical Schwarz lemma tells us that $\psi(z) = e^{i\theta}z$ for some $\theta \in [0, 2\pi)$, which implies that in fact $\varphi \in \mathrm{SU}_{1,1}(\mathbb{C})$ whence the result.

To see that the action is transitive note that

$$\begin{pmatrix}y^{1/2} & xy^{-1/2}\\ 0 & y^{-1/2}\end{pmatrix}\cdot i = x + iy$$

for any $x + iy \in \mathbb{H}^2$. Furthermore,

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot i = i \Longleftrightarrow \frac{ai+b}{ci+d} = i \Longleftrightarrow ai + b = -c + id$$

Taking real and imaginary parts, one deduces that the stabiliser of i is $\mathrm{SO}(2,\mathbb{R})$. That $G/K \to \mathbb{H}^2$ is a diffeomorphism follows from standard arguments in Differential Geometry, see Helgason II.3.2 and II.4.3(a).

(e) Set $K = \mathrm{SO}_2(\mathbb{R})$,

$$P = \left\{\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix} \mid a, b \in \mathbb{R},\ a > 0\right\},$$

$$A = \left\{\begin{pmatrix}y^{1/2} & 0\\ 0 & y^{-1/2}\end{pmatrix} \mid y \in \mathbb{R}^+\right\}, \text{ and}$$

$$N = \left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \mid x \in \mathbb{R}\right\}.$$

Prove the Iwasawa decomposition, i.e. show that

$$P \times K \longrightarrow G,\ (p,k) \longmapsto pk$$

and

$$N \times A \longrightarrow P,\ (n,a) \longmapsto na$$

are diffeomorphisms. Are these also Lie group isomorphisms?

Show that P is a semidirect product $N \rtimes A$ and that we have the diffeomorphism $N \times A \cong \mathbb{H}^2$.

This decomposition is known as the ***Iwasawa decomposition***.

Solution. It is easy to verify that $P \cap K = N \cap A = \{\mathrm{Id}\}$, and so the maps $P \times K \to G$ and $N \times A \to P$ are injective.

Consider now a matrix $g \in G$, we can consider it as determining a basis $g_i = g \cdot e_i$. Apply the Gram-Schmidt algorithm for the usual inner product on $\mathbb{R}^2$ to find a matrix $p \in P$ such that $p^{-1}g \in K$, a simple argument shows that $N \times A \to P$ is surjective. Neither of these maps are a homomorphism, however since $NA = P$,

$N \cap A = \{1\}$ and $N \lhd P$ one has $P = N \rtimes A$.

Caveat: Showing that a differentiable map is bijective does not suffice to prove that it is a diffeomorphism, however the Gram-Schmidt algorithm provides us directly with a differentiable inverse.

Finally $\mathbb{H}^2 \cong G/K \cong (N \times A \times K)/K \cong N \times A$.

(f) Prove that K is unimodular by showing that $d\mu_{\mathbb{H}^2} = y^{-2}dxdy$, $z = x+iy$, is a G-invariant volume form on the G-homogeneous space $\mathbb{H}^2$.

Solution. The derivative of $f_g : z \mapsto \frac{az+b}{cz+d}$ is

$$\frac{a(cz+d) - c(az+d)}{(cz+d)^2} = (cz+d)^{-2}$$

Write $\mathrm{Re}(f_g) = u_g$ and $\mathrm{Im}(f_g) = v_g$. The Cauchy-Riemann equations imply that

$$\begin{pmatrix} \frac{\partial u_g}{\partial x} & \frac{\partial u_g}{\partial y} \\ \frac{\partial v_g}{\partial x} & \frac{\partial y_g}{\partial x} \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(cz+d)^{-2} & -\mathrm{Im}(cz+d)^{-2} \\ \mathrm{Im}(cz+d)^{-2} & \mathrm{Re}(cz+d)^{-2} \end{pmatrix}$$

and the determinant Δ of this matrix is $|cz+d|^{-4}$. Now we calculate

$$g^*(dxdy) = \Delta dxdy = |cz+d|^{-4} \quad \text{and} \quad g^*(y^{-2}) = \mathrm{Im}\left(\frac{az+b}{cz+d}\right)^{-2} = y^{-2}|cz+d|^{-4}$$

and therefore $g^{\mu}_{\mathbb{H}^2} = \mu_{\mathbb{H}^2}$ and the assertion follows from Weil's formula.

(g) Let

$$\mathcal{F} := \{z \in \mathbb{H}^2 \mid (|z| > 1 \text{ and } -1/2 \leq \mathrm{Re}(z) < 1/2) \text{ or } (|z| = 1 \text{ and } -1/2 \leq \mathrm{Re}(z) \leq 0)\}.$$

Show that for all $z \in \mathbb{H}^2$ the orbit $G_{\mathbb{Z}}z$ intersects $\mathcal{F}$ in a unique point.
Hint: For every $G_{\mathbb{Z}}$-orbit $G_{\mathbb{Z}}z$, $z \in \mathbb{H}^2$, consider $w \in G_{\mathbb{Z}}z$ with maximal imaginary part.

Solution. Note that

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\mathrm{Im}(z)}{|cz+d|^2} \begin{cases} = \mathrm{Im}(z)|d|^{-2} & \text{if } c = 0 \\ \leq \mathrm{Im}(z)|c|^{-2} & \text{otherwise} \end{cases}$$

Hence given some $z \in \mathbb{H}^2$ the function $G_{\mathbb{Z}} \to \mathbb{H} : \gamma \mapsto \mathrm{Im}(\gamma z)$ obtains a maximum, say at $w = \gamma_0 z$. Since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ acts by translations $z \mapsto z+1$ we may assume that $-1/2 \leq \mathrm{Re}(w) < 1/2$. In addition since $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ acts as inversion $z \mapsto z^{-1}$ and $\mathrm{Im}(-1/w) = \mathrm{Im}(w)|w|^{-2}$ one clearly has $|w| \geq 1$.

It remains to show that we may impose $-1/2 \leq \mathrm{Re}(w) \leq 0$ if $|w| = 1$ and this follows from considering $z \mapsto -1/z$ again.

Thus each orbit $G_{\mathbb{Z}} \cdot z$ intersects $\mathcal{F}$, to show it is a fundamental domain it remains to show that $z, \gamma z \in \mathcal{F}$ implies $z = \gamma z$ — this follows by similar considerations to the above, and is left to the reader.

(h) Show that the volume of $\mathcal{F}$ with respect to $\mu_{\mathbb{H}^2}$ is $\pi/3$. Deduce that $\mu(G/G_{\mathbb{Z}}) < \infty$.

Solution. We can calculate

$$\begin{aligned}\int_{\mathcal{F}} d\mu_{\mathbb{H}^2} &= \int_{-1/2}^{1/2}\int_{\sqrt{1-x^2}}^{\infty} y^{-2}\,dx \\ &= \int_{-1/2}^{1/2} -y^{-1}\Big|_{y=\sqrt{1-x^2}}^{y=\infty} dydx \\ &= \int_{-1/2}^{1/2} (1-x^2)^{-1/2}\,dx \\ &= \arcsin(x)\Big|_{x=-1/2}^{x=1/2} = \pi/3.\end{aligned}$$

The second assertion follows from part (b) applied to $A = \mathcal{F}$.

2. Consider the hyperbolic n-space

$$\mathbb{H}^n = \left\{p \in \mathbb{R}^{n+1} \colon b(p,p) = -1 \text{ and } p_{n+1} > 0\right\}$$

defined by the bilinear form $b(p,q) = p_1q_1 + \ldots + p_nq_n - p_{n+1}q_{n+1}$. The tangent space at a point $p \in \mathbb{H}^n$ is defined as

$$T_p\mathbb{H}^n = \left\{x \in \mathbb{R}^{n+1} \colon \begin{array}{c}\text{There exists a smooth path } \gamma\colon (-1,1) \to \mathbb{H}^n \\ \text{such that } \gamma(0) = p \text{ and } \dot\gamma(0) = x\end{array}\right\}.$$

(a) Show that $T_p\mathbb{H}^n = \left\{x \in \mathbb{R}^{n+1} \colon b(p,x) = 0\right\}$.

Solution. Consider any $x \in T_p\mathbb{H}^n$, and let $\gamma\colon (-1,1) \to \mathbb{H}^n$ be a smooth path such that $\gamma(0) = p$ and $\dot\gamma(0) = x$. For every $t \in (-1,1)$, $b(\gamma(t),\gamma(t)) = -1$, since γ takes values in $\mathbb{H}^n$. We write $\gamma(t) = (\gamma_1(t), \cdots, \gamma_{n+1}(t))$.
Taking the derivative we see that

$$0 = \frac{d}{dt}b(\gamma(t),\gamma(t)) = \frac{d}{dt}\left(\sum_{i=1}^{n}\gamma_i(t)^2 - \gamma_{n+1}(t)^2\right) = \sum_{i=1}^{n} 2\gamma_i(t)\dot\gamma_i(t) - 2\gamma_{n+1}(t)\dot\gamma_{n+1}(t)$$

and at $t = 0$ this is

$$0 = \sum_{i=1}^{n}\gamma_i(0)\dot\gamma_i(0) - \gamma_{n+1}(0)\dot\gamma_{n+1}(0) = \sum_{i=1}^{n} p_ix_i - p_{n+1}\cdot x_{n+1} = b(p,x).$$

Thus we have shown that $T_p\mathbb{H}^n \subset \left\{x \in \mathbb{R}^{n+1} \colon b(p,x) = 0\right\}$, but since $\dim T_p\mathbb{H}^n = n$ we must have equality.

(b) Show that $g_p = b|_{T_p\mathbb{H}^n} \colon T_p\mathbb{H}^n \times T_p\mathbb{H}^n \to \mathbb{R}$ is a positive definite symmetric bilinear form on $T_p\mathbb{H}^n$ (this means that g_p is a scalar product, and $(\mathbb{H}^n, g)$ is a Riemannian manifold). *Hint: Use (a) and the Cauchy-Schwarz-inequality on* $\mathbb{R}^n$*.*

Solution. Bilinearity and symmetry $b(x, y) = b(y, x)$ follow directly, so we just need to show positive definiteness. Using (a) we can write any $p \in \mathbb{H}^n \subset \mathbb{R}^n \times \mathbb{R}$ and $x \in T_p\mathbb{H}^n \subset \mathbb{R}^n \times \mathbb{R}$ as

$$p = \left(p', \sqrt{|p'|^2 + 1}\right) \in \mathbb{H}^n \subset \mathbb{R}^n \times \mathbb{R}$$

$$x = \left(x', \frac{\langle p', x' \rangle}{\sqrt{|p'|^2 + 1}}\right) \in T_p\mathbb{H}^n \subset \mathbb{R}^n \times \mathbb{R}$$

where $\langle \cdot\ , \cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$. To show positive definiteness it remains to prove that for all $x \in T_p\mathbb{H}^n$

$$b(x, x) \geq 0.$$

with equality if and only if $x = 0$. Indeed, by the Cauchy-Schwarz-inequality

$$\langle p', x' \rangle^2 \ \leq |p'|^2|x'|^2 \leq |p'|^2|x'|^2 + |x'|^2 = (|p'|^2 + 1)|x'|^2$$

and thus

$$|x'|^2 \geq \frac{\langle p', x' \rangle^2}{|p'|^2 + 1}$$

and so

$$b(x, x) = |x'|^2 - \frac{\langle p', x' \rangle^2}{|p'|^2 + 1} \geq 0$$

with equality if and only if $x' = 0$ (which forces $x = 0$).

(c) Show that the map $s_p \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, q \mapsto -2p \cdot b(p, q) - q$ defines a well defined geodesic symmetry of $\mathbb{H}^n$ (that is, it is an involution with an isolated fixed point p). This means that the hyperbolic plane $\mathbb{H}^n$ is a Riemannian (globally) symmetric space.

Solution. To see firstly that s_p is well-defined we write as before

$$p = \left(p', \sqrt{|p'|^2 + 1}\right), \quad q = \left(q', \sqrt{|q'|^2 + 1}\right) \in \mathbb{H}^n \subset \mathbb{R}^n \times \mathbb{R}.$$

and so in particular

$$b(p, q) = \langle p', q' \rangle - \sqrt{|p'|^2 + 1}\sqrt{|q'|^2 + 1}$$

By using the definition we have that

$$\begin{aligned} s_p(q) &= -2p \cdot b(p, q) - q \\ &= \left(-2p' \cdot b(p, q) - q', -2\sqrt{|p'|^2 + 1} \cdot b(p, q) - \sqrt{|q'|^2 + 1}\right) \end{aligned}$$

and so we calculate

$$\begin{aligned} b(s_p(q), s_p(q)) &= 4|p'|^2 b(p,q)^2 + 4\langle p', q'\rangle b(p,q) + |q'|^2 \\ &\quad - \Big(4(|p'|^2+1)b(p,q)^2 + 4\sqrt{|p'|^2+1}\sqrt{|q'|^2+1}\cdot b(p,q) + |q'|^2 - 1\Big) \\ &= 4\langle p', q'\rangle b(p,q) - 4b(p,q)^2 - 4\sqrt{|p'|^2+1}\sqrt{|q'|^2+1}\cdot b(p,q) - 1 \\ &= 4b(p,q)b(p,q) - 4b(p,q)^2 - 1 = -1 \end{aligned}$$

So indeed $s_p(q) \in \mathbb{H}^n$ as required.

Note also that $s_p(p) = -2p(-1) - p = p$ is a fixed point. Next we show that s_p is an isometry, so we need to look at the differential

$$d_p s_p \colon T_pM \to T_{s_p(p)}M = T_pM.$$

If we write the points $q, p \in \mathbb{H}^n \subset \mathbb{R}^{n+1}$ in the standard basis $\{e_i\}_i$, we get the partial derivatives

$$\frac{\partial}{\partial x_i} b(p, \cdot) = \begin{cases} p_i & \text{if } i \le n \\ -p_{n+1} & \text{if } i = n+1 \end{cases}$$
$$\frac{\partial}{\partial x_i} s_p = \begin{cases} -2p \cdot p_i - e_i & \text{if } i \le n \\ 2p \cdot p_{n+1} - e_{n+1} & \text{if } i = n+1 \end{cases}$$

and thus for $v \in T_pM$ we have

$$\begin{aligned} (d_p s_p)v &= \begin{pmatrix} -2p_1^2 - 1 & -2p_1p_2 & \cdots & 2p_1p_{n+1} \\ -2p_2p_1 & -2p_2^2 - 1 & \cdots & 2p_2p_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ -2p_{n+1}p_1 & -2p_{n+1}p_2 & \cdots & 2p_{n+1}^2 - 1 \end{pmatrix} v \\ &= \begin{pmatrix} -2p_1^2v_1 - 2p_1p_2v_2 - \cdots + 2p_1p_{n+1}v_{n+1} \\ -2p_2p_1v_1 - 2p_2^2v_2 - \cdots + 2p_2p_{n+1}v_{n+1} \\ \vdots \\ -2p_{n+1}p_1v_1 - 2p_{n+1}p_2v_2 - \cdots + 2p_{n+1}^2v_{n+1} \end{pmatrix} - v \\ &= \begin{pmatrix} -2b(p,v)p_1 \\ -2b(p,v)p_2 \\ \vdots \\ -2b(p,v)p_n \end{pmatrix} - v = -v \end{aligned}$$

where we used that $b(p, v) = 0$ from part (a). Using linearity we have that

$$g_{s_p(p)}((d_p s_p)v, (d_p s_p)w) = g_p(-v, -w) = g_p(v, w),$$

so s_p is an isometry.

Now let us argue why p is an isolated fixed point, indeed suppose that $s_p(q) = q$ for some $q \in \mathbb{H}^n$. Then $-2p \cdot b(p,q) - q = q$, so $q = -b(p,q)p$, and in particular $q = \lambda p$ for λ some constant. Then $-1 = b(q,q) = b(\lambda p, \lambda p) = \lambda^2 b(p,p) = -\lambda^2$ and so $\lambda = \pm 1$. $\lambda = -1$ corresponds to $q_{n+1} < 0$, which is excluded since $\mathbb{H}^n$ is only the upper sheet of the hyperboloid, and so $q = p$ and s_p has an isolated fixed point.

By lemma II.5 of the lecture, $d_p s_p = -\operatorname{Id}_{T_p\mathbb{H}^n}$ is equivalent to $s_p \circ s_p = \operatorname{Id}_{\mathbb{H}^n}$, and so we are done. Alternatively we can calculate

$$\begin{aligned} s_p \circ s_p(q) &= s_p(-2p \cdot b(p,q) - q) \\ &= -2p \cdot b(p, -2p \cdot b(p,q) - q) - (-2p \cdot b(p,q) - q) \\ &= 4p \cdot b(p,q) b(p,p) + 2p \cdot b(p,q) + 2p \cdot b(p,q) + q = q \end{aligned}$$

3. Show that $A \mapsto gAg^t$ defines a group action of $\mathrm{SL}(n,\mathbb{R}) \ni g$ on

$$\mathcal{P}^1(n) = \left\{ A \in M_{n\times n}(\mathbb{R}) \colon A = A^t,\ \det A = 1, \quad A >> 0 \right\}.$$

Show that this action is transitive (that is, $\forall A, B \in \mathcal{P}^1(n)$ there exists some $g \in \mathrm{SL}(n,\mathbb{R})$ such that $gAg^t = B$).

You may use the Linear Algebra fact that symmetric matrices are orthogonally diagonalisable (that is, if $A = A^t$, then $\exists Q \in \mathrm{SO}(n,\mathbb{R})$ such that QAQ^t is diagonal).

Solution. We write the group action as $g.A = gAg^t$. We first need to show that the action is well defined.

- *(Symmetry)* $(g \cdot A)^T = (gAg^T)^T = gA^T g^T = gAg^T = g \cdot A$;
- *(Determinant)* $\det(g \cdot A) = \det(g)\det(A)\det(g^T) = \det(A) = 1$;
- *(Positive definite)* Let $x \in \mathbb{R}^n \backslash \{0\}$, then

  $$x^T(g \cdot A)x = x^T gAg^T x = (g^T x)^T A (g^T x) > 0$$

  since $g^T x \in \mathbb{R}^n \backslash \{0\}$.

We also note that it is a group action — clearly $\operatorname{Id} \cdot A = A$, and $(gh) \cdot A = ghAh^T g^T = g \cdot (h \cdot A)$.

It remains to show that the action is transitive. Let $A, B \in \mathcal{P}^1(n)$. We can use linear algebra to get $Q, R \in \mathrm{SO}(n) < \mathrm{SL}(n,\mathbb{R})$ such that $Q.A$ and $R.B$ are diagonal, have determinant 1 and are positive definite (by the well-definedness of the group action). Positive definiteness implies that all entries are non-negative. Then the matrix $\Lambda = (Q \cdot A)(R \cdot B)^{-1}$ is also diagonal, has determinant 1 and positive elements on the diagonal. We can therefore take the component-wise square root $\sqrt{\Lambda}$ of Λ.

Set $g = Q^{-1}\sqrt{\Lambda}R \in \mathrm{SL}(n,\mathbb{R})$ and use the fact that $R.B$ commutes with $\sqrt{\Lambda}$ since they are diagonal to see that

$$g \cdot B = Q^{-1}\sqrt{\Lambda}R \cdot B = Q^{-1} \cdot \left(\sqrt{\Lambda}(R \cdot B)\sqrt{\Lambda}^T\right) = Q^{-1}. \left(\sqrt{\Lambda}\sqrt{\Lambda}^T R \cdot B\right)$$
$$= Q^{-1} \cdot (\Lambda R \cdot B) = Q^{-1} \cdot ((Q \cdot A)(R \cdot B)^{-1}(R \cdot B)) = Q^{-1} \cdot Q \cdot A = A.$$

this shows that from any point $B \in \mathcal{P}^1(n)$ you can go to any point $A \in \mathcal{P}^1(n)$ by the action of $\mathrm{SL}(n, \mathbb{R})$ (the action is transitive).