Solutions Exercise Sheet 1

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1. Let $G = \mathrm{SL}_2(\mathbb{R})$. The aim of this exercise is to show that $G_{\mathbb{Z}} = \mathrm{SL}_2(\mathbb{Z})$ is a lattice in G.

   (a) Argue that $G_{\mathbb{Z}}$ is discrete in G and that both G and $G_{\mathbb{Z}}$ are unimodular.

   From this we know that $G/G_{\mathbb{Z}}$ admits a nonzero G-invariant measure μ which is unique up to a non-zero constant. In order to show that $G_{\mathbb{Z}}$ is a lattice we have to show that $\mu(G/G_{\mathbb{Z}}) < \infty$. For this, we will use the following fact:

   (b) Assume that there exists a measurable set $A \subseteq G$ of finite measure such that every $G_{\mathbb{Z}}$-orbit intersects A (that is, for every $g \in G$ there exists some $\gamma \in G_{\mathbb{Z}}$ such that $g\gamma \in A$). Show that $\mu(G/G_{\mathbb{Z}})$ is finite.

   (c) Show that the map sending

   $$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \text{ to } z \longmapsto g \cdot z := \frac{az+b}{cz+d}$$

   is a group homomorphism $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{Bih}(\mathbb{H}^2)$, where $\mathrm{Bih}(\mathbb{H}^2)$ denotes the biholomorphic maps of the complex upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$. Show that its kernel is $\{\pm I\}$ where I denotes as usual the 2×2 identity matrix.

   These maps are known as *Möbius transformations*.

   (d) Prove that the induced homomorphism

   $$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) / \{\pm I\} \to \mathrm{Bih}(\mathbb{H}^2)$$

   of (c) is actually an isomorphism. For the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}^2$ from above determine the orbit Gi and stabilizer K of $i \in \mathbb{H}^2$. (Show also that K is compact.) Using this, show that we have a diffeomorphism

   $$G/K \longrightarrow \mathbb{H}^2,\ g \longmapsto g \cdot i.$$

   (e) Set $K = \mathrm{SO}_2(\mathbb{R})$,

   $$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathbb{R},\, a > 0 \right\},$$

   $$A = \left\{ \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \mid y \in \mathbb{R}^+ \right\}, \text{ and}$$

   $$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}.$$

Prove the Iwasawa decomposition, i.e. show that

$$P \times K \longrightarrow G,\ (p,k) \longmapsto pk$$

and

$$N \times A \longrightarrow P,\ (n,a) \longmapsto na$$

are diffeomorphisms. Are these also Lie group isomorphisms?
Show that P is a semidirect product $N \rtimes A$ and that we have the diffeomorphism $N \times A \cong \mathbb{H}^2$.

This decomposition is known as the ***Iwasawa decomposition****.*

(f) Prove that K is unimodular by showing that $d\mu_{\mathbb{H}^2} = y^{-2}dxdy$, $z = x+iy$, is a G-invariant volume form on the G-homogeneous space $\mathbb{H}^2$.

(g) Let

$$\mathcal{F} := \{z \in \mathbb{H}^2 \mid (|z| > 1 \text{ and } -1/2 \le \mathrm{Re}(z) < 1/2) \text{ or } (|z| = 1 \text{ and } -1/2 \le \mathrm{Re}(z) \le 0)\}.$$

Show that for all $z \in \mathbb{H}^2$ the orbit $G_{\mathbb{Z}}z$ intersects $\mathcal{F}$ in a unique point.
Hint: For every $G_{\mathbb{Z}}$-orbit $G_{\mathbb{Z}}z$, $z \in \mathbb{H}^2$, consider $w \in G_{\mathbb{Z}}z$ with maximal imaginary part.

(h) Show that the volume of $\mathcal{F}$ with respect to $\mu_{\mathbb{H}^2}$ is $\pi/3$. Deduce that $\mu(G/G_{\mathbb{Z}}) < \infty$.

2. Consider the hyperbolic n-space

$$\mathbb{H}^n = \{p \in \mathbb{R}^{n+1} \colon b(p,p) = -1 \text{ and } p_{n+1} > 0\}$$

defined by the bilinear form $b(p,q) = p_1q_1 + \ldots + p_nq_n - p_{n+1}q_{n+1}$. The tangent space at a point $p \in \mathbb{H}^n$ is defined as

$$T_p\mathbb{H}^n = \left\{ x \in \mathbb{R}^{n+1} \colon \begin{array}{c} \text{There exists a smooth path } \gamma\colon (-1,1) \to \mathbb{H}^n \\ \text{such that } \gamma(0) = p \text{ and } \dot\gamma(0) = x \end{array} \right\}.$$

(a) Show that $T_p\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \colon b(p,x) = 0\}$.

(b) Show that $g_p = b|_{T_p\mathbb{H}^n} \colon T_p\mathbb{H}^n \times T_p\mathbb{H}^n \to \mathbb{R}$ is a positive definite symmetric bilinear form on $T_p\mathbb{H}^n$ (this means that g_p is a scalar product, and $(\mathbb{H}^n, g)$ is a Riemannian manifold).
Hint: Use (a) and the Cauchy-Schwarz-inequality on $\mathbb{R}^n$*.*

(c) Show that the map $s_p \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, q \mapsto -2p \cdot b(p,q) - q$ defines a well defined geodesic symmetry of $\mathbb{H}^n$ (that is, it is an involution with an isolated fixed point p). This means that the hyperbolic plane $\mathbb{H}^n$ is a Riemannian (globally) symmetric space.

3. Show that $A \mapsto gAg^t$ defines a group action of $\mathrm{SL}(n,\mathbb{R}) \ni g$ on

$$\mathcal{P}^1(n) = \{A \in M_{n\times n}(\mathbb{R}) \colon A = A^t,\ \det A = 1,\quad A >> 0\}.$$

Show that this action is transitive (that is, $\forall A, B \in \mathcal{P}^1(n)$ there exists some $g \in \mathrm{SL}(n,\mathbb{R})$ such that $gAg^t = B$).

You may use the Linear Algebra fact that symmetric matrices are orthogonally diagonalisable (that is, if $A = A^t$*, then* $\exists Q \in \mathrm{SO}(n,\mathbb{R})$ *such that* QAQ^t *is diagonal).*