Exercise Sheet 2

Exercise 1 (Compact Lie groups as symmetric spaces)**.** Let G be a compact connected Lie group and let

$$G^* = \{(g, g) \in G \times G : g \in G\} < G \times G$$

denote the diagonal subgroup.

(a) Show that the pair $(G \times G, G^*)$ is a Riemannian symmetric pair, and the coset space $G \times G/G^*$ is diffeomorphic to G.

> **Solution.** Consider the mapping $\sigma\colon (g_1, g_2) \mapsto (g_2, g_1)$. This is an involutive automorphism of the product group $G \times G$. The fixed set of σ is precisely the diagonal G^*. It follows that the pair $(G \times G, G^*)$ is a Riemannian symmetric pair. The coset space $G \times G/G^*$ is diffeomorphic to G under the mapping $[(g_1, g_2)] \mapsto \pi(g_1, g_2) \coloneqq g_1 g_2^{-1}$.

(b) Using the above, explain how any compact connected Lie group G can be regarded as a Riemannian globally symmetric space.

> **Solution.** By Proposition 3.4 from Helgason, Ch. IV, we see that G is a Riemanian globally symmetric space in each bi-invariant Riemannian structure; note here that a Riemannian structure on $G \times G/G^*$ is $G \times G$-invariant if and only if the corresponding Riemannian structure on G is bi-translation invariant.

(c) Let $\mathfrak{g}$ denote the Lie algebra of G. Show that the exponential map from $\mathfrak{g}$ into the Lie group G coincides with the *Riemannian* exponential map from $\mathfrak{g}$ into the Riemannian globally symmetric space G.

> **Solution.** Note that the product algebra $\mathfrak{g} \times \mathfrak{g}$ is the Lie algebra of $G \times G$. Let $\exp^*$ denote the exponential mapping of $\mathfrak{g} \times \mathfrak{g}$ int $G \times G$, exp denote the exponential mapping of $\mathfrak{g}$ into G, and Exp denote the Riemannian exponential mapping of $\mathfrak{g} \cong T_e G$ into G (considered as a Riemannian globally symmetric space). We want to show that $\exp X = \operatorname{Exp} X$ for all $X \in \mathfrak{g}$. Using $d\pi(X, Y) = X - Y$, we deduce that $\pi(\exp^*(X, -X)) = \operatorname{Exp}(d\pi(X, -X))$. Hence $\exp X \cdot (\exp(-X))^{-1} = \operatorname{Exp}(2X)$ and this implies that $\exp X = \operatorname{Exp} X$.

Exercise 2 (Compact semisimple Lie groups as symmetric spaces)**.** A compact semisimple Lie group G has a bi-invariant Riemannian structure Q such that Q_e is the negative of the Killing form of the Lie algebra $\mathfrak{g} = \operatorname{Lie}(G)$. If G is considered as a symmetric space $G \times G/G^*$ as in the above exercise, it acquires a bi-invariant Riemannian structure Q^* from the Killing form of $\mathfrak{g} \times \mathfrak{g}$. Show that $Q = 2Q^*$.

Solution. Let π and σ be as in the above solution. The map $d\pi$ maps the -1 eigenspace of $d\sigma$ onto $\mathfrak{g}$ as follows: $d\pi(X,-X) = 2X$. Using this, we can check that

$$2B_{\mathfrak{g}\times\mathfrak{g}}((X,-X),(X,-X)) = B_{\mathfrak{g}}(2X,2X),$$

which is equivalent to $Q = 2Q^*$.

Exercise 3 (Closed differential forms)**.** Let M be a Riemannian globally symmetric space and let ω be a differential form on M invariant under $\mathrm{Isom}(M)^\circ$. Prove that $d\omega = 0$.

Solution. Let s_m denote the geodesic symmetry at some point $m \in M$, and let $\omega \in \Omega^p(M)$ be an invariant differential p-form on M. Because $d_m s_m = -\mathrm{Id}\colon T_pM \to T_pM$, we get $(s_m^*\omega)_m = (-1)^p\omega_m$ at the point $m \in M$. Because ω is invariant, $s_m^*\omega$ is invariant as well. Because $\mathrm{Iso}(M)^\circ$ acts transitively, invariant differential forms are determined by their value at a single point such that

$$s_m^*\omega = (-1)^p\omega$$

on all of M.

Therefore, we obtain

$$d\omega = (-1)^p d(s_m^*\omega) = (-1)^p s_m^* d\omega = (-1)^{2p+1} d\omega,$$

whence $d\omega = 0$.

Exercise 4 (A symmetric space with non-compact K)**.** Let $G = \widetilde{\mathrm{SL}(2,\mathbb{R})}$ and $K = \widetilde{\mathrm{SO}(2,\mathbb{R})}$. The aim of this exercise is to show that (G,K) is a symmetric pair with non-compact K.

(a) Prove that $\sigma\colon \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}), g \mapsto {}^tg^{-1}$ is an involution.

Solution. σ is a homomorphism, and $\sigma^2 = \mathrm{Id}$, by standard properties of transpose and inverse, the latter also shows that σ is a bijection. Also most matrices in $\mathrm{SL}(2,\mathbb{R})$ aren't fixed by σ, so in particular $\sigma \neq \mathrm{Id}$.

(b) By covering space theory we can lift σ to the universal cover G. Prove that $\tilde{\sigma}\colon G \to G$ is an involution as well. You may use that the universal cover of a path-connected topological group is again a topological group.

Solution. Recall from covering space theory the following fact:

Let $\pi : C \to X$ be a cover and $f : Y \to X$ a continuous map. Pick $y_0 \in Y$ and $c_0 \in C$, which lies over $f(y_0)$ (that is, $\pi(c_0) = f(y_0)$). If Y is simply connected, then there exists a unique lift $\tilde{f} : Y \to C$ with $\pi \circ \tilde{f} = f$ and $\tilde{f}(y_0) = c_0$.

In our case, $Y = C = G$ is the universal cover, and thus is simply connected. Let us write $\pi : G \to \mathrm{SL}(2,\mathbb{R})$ and $f = \sigma \circ \pi$. Fix an element $\tilde{\mathrm{Id}}$ in the universal cover with $\pi(\tilde{\mathrm{Id}}) = \mathrm{Id}$, then we get a unique map $\tilde{\sigma} : G \to G$ with $\tilde{\sigma}(\tilde{\mathrm{Id}}) = \tilde{\mathrm{Id}}$.

We now have to show that $\tilde{\sigma}$ is a homomorphism, so consider the map

$$\varphi : G \times G \to G : (g,h) \mapsto \tilde{\sigma}(gh)^{-1}\tilde{\sigma}(g)\tilde{\sigma}(h)$$

Since $\pi(gh) = \pi(g)\pi(h)$ (the multiplication in the universal covering is the lift of the multiplication in the group), π is a homomorphism. We have

$$\begin{aligned}
\pi(\varphi(g,h)) &= \pi(\tilde{\sigma}(gh)^{-1}\tilde{\sigma}(g)\tilde{\sigma}(h)) \\
&= \pi(\tilde{\sigma}(gh)^{-1})\pi(\tilde{\sigma}(g))\pi(\tilde{\sigma}(h)) \\
&= \pi(\tilde{\sigma}(gh))^{-1}\pi(\tilde{\sigma}(g))\pi(\tilde{\sigma}(h)) \\
&= \sigma(\pi(gh))^{-1}\sigma(\pi(g))\sigma(\pi(h)) \\
&= \sigma(\pi(g)\pi(h))^{-1}\sigma(\pi(g)\sigma(\pi(h)) \\
&= \mathrm{Id} = \tilde{\mathrm{Id}}
\end{aligned}$$

So φ is a lift of $\pi \circ \varphi$, as is the constant function $(g,h) \mapsto \tilde{\mathrm{Id}}$. By uniqueness of the lift we conclude that $\varphi(g,h) = \tilde{\mathrm{Id}}$, that is $\tilde{\sigma}$ is a homomorphism.

Now note that $\pi \circ \tilde{\sigma} \circ \tilde{\sigma} = \sigma \circ \pi \circ \tilde{\sigma} = \sigma \circ \sigma \circ \pi = \pi$, so $\tilde{\sigma} \circ \tilde{\sigma}$ as well as the constant function $g \mapsto \tilde{\mathrm{Id}}$ are lifts of π, and so by uniqueness of lifts we obtain that $\tilde{\sigma} \circ \tilde{\sigma}(g) = \tilde{\mathrm{Id}}$ for all $g \in G$.

Finally, since σ is not the identity, $\tilde{\sigma}$ is not a lift of the identity, and so isn't the identity map.

(c) Prove that $G^{\tilde{\sigma}} = K \cong \mathbb{R}$.

Solution. The map $\sigma|_{\mathrm{SO}(2)} : \mathrm{SO}(2) \to \mathrm{SO}(2)$ is the identity. The lift

$$\tilde{\sigma}|_{\widetilde{\mathrm{SO}(2)}} : \widetilde{\mathrm{SO}(2)} \to \widetilde{\mathrm{SO}(2)}$$

therefore also has to be the identity, by uniqueness of the lift. So if $g \in K = \widetilde{\mathrm{SO}(2)}$, then $\tilde{\sigma}(g) = g$, and $g \in G^{\tilde{\sigma}}$.

(d) Prove that $\mathrm{Ad}_G(K) = \mathrm{Ad}_{\mathrm{SL}(2,\mathbb{R})}(\mathrm{SO}(2,\mathbb{R}))$.

Solution. The Lie algebra $\mathfrak{g}$ only depends on a neighbourhood, so

$$\mathrm{Lie}(\mathrm{SL}(2,\mathbb{R})) = \mathfrak{g} = \mathrm{Lie}(\widetilde{\mathrm{SL}(2,\mathbb{R})})$$

Since the left-multiplication on the universal cover is the lift of the left-multiplication of $\mathrm{SL}(2,\mathbb{R})$, they can be identified in a small neighbourhood around $o = \mathrm{Id}$. The adjoint representation $\mathrm{Ad}(g) = d_o c_g$ is a derivative at a point, and thus only depends on a neighbourhood. We conclude that the images of the adjoint representations are equal.

(e) Show that $\mathrm{Ad}_{\mathrm{SL}(2,\mathbb{R})}(\mathrm{SO}(2,\mathbb{R})) \simeq \mathrm{SO}(2,\mathbb{R})/\{\pm 1\}$.

Solution. The elements g in the kernel satisfy $X = gXg^{-1}$ for all

$$X \in \mathfrak{sl}(2,\mathbb{R}) = \{X \in \mathbb{R}^{2\times 2} \mid \mathrm{Tr}(X) = 0\}$$

Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $X = \left(\begin{smallmatrix} x & y \\ z & -x \end{smallmatrix}\right)$ then we calculate

$$Xg = \begin{pmatrix} ax + cy & bx + dy \\ az - cx & bz - dx \end{pmatrix} = \begin{pmatrix} ax + bz & ay - bx \\ cx + dz & cy - dx \end{pmatrix} = gX$$

so $bz = cy$ for all $z, y \in \mathbb{R}$, so $b = c = 0$. Hence we have $ay = dy$ and $dz = az$ which imply $a = d$.

Since $g \in \mathrm{SO}(2)$ we have $\det(g) = a^2 = 1$ and so $a = d = \pm 1$, and $g = \pm\,\mathrm{Id}$ (and indeed both are in $\mathrm{SO}(2)$).

Exercise 5. (a) Let G be a connected topological group and $N \lhd G$ a normal subgroup which is discrete. Show that $N \subset Z(G)$ is contained in the center $Z(G)$ of G.

(b) Let (G, K) be a Riemannian symmetric pair and $Z(G)$ the center of G. Show that $\mathrm{Ad}_G\colon G \to \mathrm{GL}(\mathfrak{g})$ induces an isomorphism of Lie groups:

$$K/(K \cap Z(G)) \to \mathrm{Ad}_G(K) < \mathrm{GL}(\mathfrak{g}).$$

Done in Lie Groups - contact if unsure.