

Exercise Sheet 2

This requires material that will be covered in week 4, and so the submission deadline has been extended to Saturday 19th October.

Exercise 1 (Compact Lie groups as symmetric spaces). Let G be a compact connected Lie group and let

$$G^* = \{(g, g) \in G \times G : g \in G\} < G \times G$$

denote the diagonal subgroup.

- (a) Show that the pair $(G \times G, G^*)$ is a Riemannian symmetric pair, and the coset space $G \times G/G^*$ is diffeomorphic to G .
- (b) Using the above, explain how any compact connected Lie group G can be regarded as a Riemannian globally symmetric space.
- (c) Let \mathfrak{g} denote the Lie algebra of G . Show that the exponential map from \mathfrak{g} into the Lie group G coincides with the *Riemannian* exponential map from \mathfrak{g} into the Riemannian globally symmetric space G .

Exercise 2 (Compact semisimple Lie groups as symmetric spaces). A compact semisimple Lie group G has a bi-invariant Riemannian structure Q such that Q_e is the negative of the Killing form of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$. If G is considered as a symmetric space $G \times G/G^*$ as in the above exercise, it acquires a bi-invariant Riemannian structure Q^* from the Killing form of $\mathfrak{g} \times \mathfrak{g}$. Show that $Q = 2Q^*$.

Exercise 3 (Closed differential forms). Let M be a Riemannian globally symmetric space and let ω be a differential form on M invariant under $\text{Isom}(M)^\circ$. Prove that $d\omega = 0$.

Exercise 4 (A symmetric space with non-compact K). Let $G = \widetilde{\text{SL}(2, \mathbb{R})}$ and $K = \widetilde{\text{SO}(2, \mathbb{R})}$. The aim of this exercise is to show that (G, K) is a symmetric pair with non-compact K .

- (a) Prove that $\sigma: \text{SL}(2, \mathbb{R}) \rightarrow \text{SL}(2, \mathbb{R}), g \mapsto {}^t g^{-1}$ is an involution.
- (b) By covering space theory we can lift σ to the universal cover G . Prove that $\tilde{\sigma}: G \rightarrow G$ is an involution as well. You may use that the universal cover of a path-connected topological group is again a topological group.
- (c) Prove that $G^{\tilde{\sigma}} = K \cong \mathbb{R}$.
- (d) Prove that $\text{Ad}_G(K) = \text{Ad}_{\text{SL}(2, \mathbb{R})}(\text{SO}(2, \mathbb{R}))$.
- (e) Show that $\text{Ad}_{\text{SL}(2, \mathbb{R})}(\text{SO}(2, \mathbb{R})) \simeq \text{SO}(2, \mathbb{R})/\{\pm 1\}$.

Exercise 5. (a) Let G be a connected topological group and $N \triangleleft G$ a normal subgroup which is discrete. Show that $N \subset Z(G)$ is contained in the center $Z(G)$ of G .

(b) Let (G, K) be a Riemannian symmetric pair and $Z(G)$ the center of G . Show that $\text{Ad}_G: G \rightarrow \text{GL}(\mathfrak{g})$ induces an isomorphism of Lie groups:

$$K/(K \cap Z(G)) \rightarrow \text{Ad}_G(K) < \text{GL}(\mathfrak{g}).$$