Exercise Sheet 3

Exercise 1. Consider $G = \mathrm{SO}(1,n)^\circ$ with the involutive Lie group automorphism

$$\sigma : G \to G, g \mapsto J_n g J_n$$

where

$$J_n = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \in \mathrm{SO}(1,n).$$

Further let

$$K = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(n) \end{pmatrix} \cong \mathrm{SO}(n).$$

It can be shown that (G, K, σ) is a Riemannian symmetric pair and that G/K is isometric to $\mathbb{H}^n$.

(a) Show that $\Theta = d\sigma : \mathfrak{g} \to \mathfrak{g}$ takes the form

$$\Theta(X) = \begin{pmatrix} 0 & -x^t \\ -x & D \end{pmatrix}$$

for all

$$X = \begin{pmatrix} 0 & x^t \\ x & D \end{pmatrix} \in \mathfrak{g} = \mathfrak{so}(1,n).$$

Deduce that

$$\mathfrak{p} = E_{-1}(\Theta) = \left\{ \begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix} : x \in \mathbb{R}^n \right\},$$
$$\mathfrak{k} = E_1(\Theta) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} : D \in \mathfrak{so}(n) \right\} \cong \mathfrak{so}(n).$$

Solution. We compute

$$\Theta(X) = J_n X J_n = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & x^t \\ x & D \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -x^t \\ x & D \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} 0 & -x^t \\ -x & D \end{pmatrix}$$

for all $X = \begin{pmatrix} 0 & x^t \\ x & D \end{pmatrix} \in \mathfrak{so}(1,n)$. Thus $\Theta(X) = -X$ implies $D = 0$, and likewise $\Theta(X) = X$ implies $x = 0$.

(b) Let $\pi : G \to G/K$ denote the usual quotient map and set $\overline{X} := d_e\pi(X) \in T_o(G/K)$ for all $X \in \mathfrak{g}$. Further let $\langle X, Y\rangle := \frac{1}{2}\mathrm{tr}(XY)$ for all $X, Y \in \mathfrak{p}$.

Show that

$$R_o(\overline{X}, \overline{Y})\overline{Z} = \langle X, Z\rangle\overline{Y} - \langle Y, Z\rangle\overline{X}$$

for all $X, Y, Z \in \mathfrak{p}$. Deduce that G/K has constant sectional curvature -1.

Hint: You may use the following formula without proof:

The Riemann curvature tensor at $o \in M = G/K$ is given by

$$R_o(\overline{X}, \overline{Y})\overline{Z} = -\overline{[[X, Y], Z]}$$

for all $\overline{X}, \overline{Y}, \overline{Z} \in T_oM$.

Solution. Let

$$X = \begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & y^t \\ y & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & z^t \\ z & 0 \end{pmatrix} \in \mathfrak{p}.$$

Note that

$$\langle X, Y\rangle = \tfrac{1}{2}\mathrm{tr}\left(\begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix}\begin{pmatrix} 0 & y^t \\ y & 0 \end{pmatrix}\right) = \tfrac{1}{2}\mathrm{tr}\begin{pmatrix} x^ty & 0 \\ 0 & xy^t \end{pmatrix} = \langle x, y\rangle,$$

where the latter is to be understood as the Euclidean inner product of the vectors $x, y \in \mathbb{R}^n$.

By the given formula we obtain

$$R_o(\overline{X}, \overline{Y})\overline{Z} = -\overline{[[X, Y], Z]}.$$

First, we compute

$$\begin{aligned} [X, Y] &= \begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix}\begin{pmatrix} 0 & y^t \\ y & 0 \end{pmatrix} - \begin{pmatrix} 0 & y^t \\ y & 0 \end{pmatrix}\begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix} \\ &= \begin{pmatrix} x^ty & 0 \\ 0 & xy^t \end{pmatrix} - \begin{pmatrix} y^tx & 0 \\ 0 & yx^t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & xy^t - yx^t \end{pmatrix}, \end{aligned}$$

and then

$$\begin{aligned} [[X, Y], Z] &= \begin{pmatrix} 0 & 0 \\ 0 & xy^t - yx^t \end{pmatrix}\begin{pmatrix} 0 & z^t \\ z & 0 \end{pmatrix} - \begin{pmatrix} 0 & z^t \\ z & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & xy^t - yx^t \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ (xy^t - yx^t)z & 0 \end{pmatrix} - \begin{pmatrix} 0 & z^t(xy^t - yx^t) \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \langle y, z\rangle x^t - \langle x, z\rangle y^t \\ \langle y, z\rangle x - \langle x, z\rangle y & 0 \end{pmatrix} \\ &= \langle Y, Z\rangle X - \langle X, Z\rangle Y. \end{aligned}$$

This implies our claim.

As for the sectional curvature, let $V \subset \mathfrak{p}$ be a two-dimensional linear subspace and choose $X, Y \in \mathfrak{p}$ to be an orthonormal basis of V. Then

$$\kappa_o(\overline{V}) = R_o(\overline{X}, \overline{Y}, \overline{Y}, \overline{X}) = \langle R_o(\overline{X}, \overline{Y})\overline{Y}, \overline{X}\rangle = \langle \langle X, Y\rangle \overline{Y} - \langle Y, Y\rangle \overline{X}, \overline{X}\rangle = -1.$$

(c) Compute that

$$\exp\left(t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

for all $t \in \mathbb{R}$.

Solution. Note that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{2k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{2k+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for all $k \in \mathbb{N}$. Thus

$$\begin{aligned} \exp\left(t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n \\ &= \sum_{k=0}^{\infty} \frac{t^{2k}}{2k!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{2k} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{2k+1} \\ &= \cosh t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sinh t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \end{aligned}$$

for all $t \in \mathbb{R}$.

Exercise 2 (Closed adjoint subgroups of $\mathrm{SL}_n(\mathbb{R})$ and their symmetric spaces)**.** Consider the Riemannian symmetric pair (G, K, σ) where $G = \mathrm{SL}_n(\mathbb{R})$, $K = \mathrm{SO}(n, \mathbb{R})$ and $\sigma : \mathrm{SL}_n(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R}), g \mapsto {}^t(g^{-1})$. Further let $H \leq G$ be a closed, connected subgroup that is adjoint, i.e. it is closed under transposition $h \mapsto {}^t h$.

(a) Show that $(H, H \cap K, \sigma|_H)$ is again a Riemannian symmetric pair.

Solution. Note that $H = \sigma(\sigma(H)) \subseteq \sigma(H)$ since $\sigma : G \to G$ is an involutive Lie group automorphism. Because $H \leq G$ is a closed subgroup and hence an embedded submanifold σ restricts to an involutive Lie group automorphism $\sigma|_H : H \to H$.

Let us check that $(H^\sigma)^\circ \subseteq H \cap K \subseteq H^\sigma$. Concerning the first inclusion note that $(H^\sigma)^\circ \subseteq H^\sigma \subseteq G^\sigma$ whence $(H^\sigma)^\circ \subseteq (G^\sigma)^\circ$. Then the first inclusion follows from $(H^\sigma)^\circ \subseteq H$ and $(G^\sigma)^\circ \subseteq K$. The second inclusion follows easily from the fact that $H^\sigma = H \cap G^\sigma$ and $K \subseteq G^\sigma$.

Finally, $K = \mathrm{SO}(n, \mathbb{R})$ is compact whence the intersection $H \cap K$ is also compact as is the image $\mathrm{Ad}_H(H \cap K) \leq \mathrm{GL}(\mathfrak{h})$.

(b) Show that $i : H \hookrightarrow G$ descends to a smooth embedding $\varphi : H/H \cap K \hookrightarrow G/K$ such that its image is a totally geodesic submanifold of G/K.

Solution. Denote by $\pi : G \to G/K$ and $\pi' : H \to H/H \cap K$ the usual quotient maps. Define

$$\varphi : H/H \cap K \to G/K, \pi'(h) \mapsto \pi(i(h)).$$

That map is well-defined because

$$\begin{aligned} \pi'(h_1) = \pi'(h_2) &\iff h_2^{-1}h_1 \in H \cap K \\ &\iff i(h_2^{-1}h_1) \in K \\ &\iff \pi(i(h_1)) = \pi(i(h_2)) \end{aligned}$$

for all $h_1, h_2 \in H$. This argument also shows that φ is injective whence it is a bijection onto its image $N := \mathrm{im}\varphi$. We obtain the following commutative diagram.

$$\begin{array}{ccc} H & \overset{i}{\hookrightarrow} & G \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ H/H \cap K & \overset{\varphi}{\hookrightarrow} & N \subseteq G/K \end{array}$$

Clearly, φ is continuous by the universal property of the quotient topology. We will now show that $\varphi : H/H \cap K \to N$ is proper, i.e. preimages of compact sets are compact. That will prove that φ is actually open onto its image because proper continuous maps are closed and continuous closed bijections are open. Let $C \subseteq N \subset G/K$ be compact. Because G is locally compact there is a compact set $C' \subseteq G$ such that $\pi(C') = C$ and

$$\pi^{-1}(C) = \pi^{-1}(\pi(C')) = \bigcup_{k \in K} C'k$$

is compact, i.e. π is proper. Further $i^{-1}(\pi^{-1}(C)) = H \cap \pi^{-1}(C)$ is compact since H is closed. Finally, $\varphi^{-1}(C) = \pi'(i^{-1}(\pi^{-1}(C)))$ is compact since π' is continuous. Therefore, $\varphi : H/H \cap K \to N$ is a homeomorphism.

Note that the smooth structure on $H/H \cap K$ is such that π' is a smooth submersion whence $\varphi : H/H \cap K \to G/K$ is smooth because $\pi \circ i : H \to G/K$ is smooth. Also, φ is equivariant with respect to the respective H-actions on $H/H \cap K$ and N. Because H acts transitively on $H/H \cap K$ it is easy to see that φ has constant rank. By the global rank theorem [**?**, Theorem 4.14] $\varphi : H/H \cap K \to N$ is a smooth immersion. This shows that φ is a smooth embedding.

In order to check that N is a totally geodesic submanifold we will show that its tangent space amounts to a Lie triple system $\mathfrak{n} \subseteq \mathfrak{p}$. Let $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{g} = \mathrm{Lie}(G)$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Define $\Theta|_H := d\sigma|_H : \mathfrak{h} \to \mathfrak{h}$. It is easy to check that the corresponding Cartan decomposition is just $\mathfrak{h} = \mathfrak{k}' \oplus \mathfrak{p}'$ with $\mathfrak{k}' := \mathfrak{k} \cap \mathfrak{h}$ and $\mathfrak{p}' := \mathfrak{p} \cap \mathfrak{h}$. As we know $d_e\pi' : \mathfrak{p}' \to T_o H/H \cap K$ is an isomorphism as is $d_o\varphi : T_o H/H \cap K \to T_o N$. By commutativity of the above diagram $T_o N = d_e\pi(\mathfrak{n})$ where $\mathfrak{n} = di(\mathfrak{p}') \subseteq \mathfrak{p}$. The subspace $\mathfrak{n}$ is a Lie triple system since $di : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism

$$[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] = [[di(\mathfrak{p}'), di(\mathfrak{p}')], di(\mathfrak{p}')] = di([[\mathfrak{p}', \mathfrak{p}'], \mathfrak{p}']) \subseteq di([\mathfrak{k}', \mathfrak{p}']) \subseteq di(\mathfrak{p}') = \mathfrak{n}.$$

Thus N is a totally geodesic submanifold.

Exercise 3 (The symplectic group $\mathrm{Sp}(2n, \mathbb{R})$)**.** Let $H = \mathrm{Sp}(2n, \mathbb{R}) = \{g \in \mathrm{GL}_{2n}(\mathbb{R}) : g^t J g = J\}$ be the symplectic group, where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

(a) Show that $\mathrm{Sp}(2n, \mathbb{R}) \leq \mathrm{SL}(2n, \mathbb{R}) =: G$ is a closed connected *adjoint* subgroup of G.

What can we deduce from exercise 2 about $(H, H \cap K, \sigma|_H)$?

Solution. We will only show that $\mathrm{Sp}(2n, \mathbb{R})$ is adjoint. Let $g \in \mathrm{Sp}(2n, \mathbb{R})$. Note that $J^{-1} = -J = J^t$. Then

$$g^t J g = J \implies g^t = -J g^{-1} J$$

and thus

$$g J g^t = g J(-J g^{-1} J) = g g^{-1} J = J$$

whence $g^t \in \mathrm{Sp}(2n, \mathbb{R})$.

Now set $K' := \mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{SO}(2n, \mathbb{R})$. By exercise 2 H/K' is again a symmetric space and the inclusion $H \hookrightarrow \mathrm{SL}(2n, \mathbb{R})$ descends to a smooth embedding with image a totally geodesic submanifold of $\mathrm{SL}(2n, \mathbb{R})/\mathrm{SO}(2n, \mathbb{R})$.

(b) Denote by $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ the standard symplectic form given by $\omega(x, y) = x^t J y$.

Show that $B : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}, (x, y) \mapsto \omega(Jx, y)$ is a symmetric positive definite bilinear form.

Solution. Let $x, y \in \mathbb{R}^{2n}$. Then

$$B(x, y) = \omega(Jx, y) = (Jx)^t J y = -x^t J^2 y = x^t y = \langle x, y \rangle.$$

(c) An endomorphism $M \in \mathrm{End}(\mathbb{R}^{2n})$ is called a complex structure if $M^2 = -\mathrm{Id}$. We say that M is ω-compatible if $(x, y) \mapsto \omega(Mx, y)$ is a symmetric positive definite bilinear form. Denote the set of all ω-compatible complex structures by S_{2n}.

Show that $H = \mathrm{Sp}(2n, \mathbb{R})$ acts on S_{2n} via conjugation and deduce that there is a bijection $S_{2n} \cong H/H \cap K$.

Solution. Let $M \in S_{2n}$ and $g \in \mathrm{Sp}(2n, \mathbb{R})$. Then

$$(gMg^{-1})^2 = gM^2g^{-1} = -gg^{-1} = -\mathrm{Id}.$$

If we denote by $B_M(x, y) = \omega(Mx, y)$ the associated symmetric positive definite bilinear form then

$$\begin{aligned} B_{gMg^{-1}}(x, y) &= \omega(gMg^{-1}x, y) = \omega(Mg^{-1}x, g^{-1}y) \\ &= B_M(g^{-1}x, g^{-1}y) = g_* B_M(x, y) \end{aligned}$$

for all $x, y \in \mathbb{R}^{2n}$, which is again a symmetric positive definite bilinear form.

Therefore, $\mathrm{Sp}(2n, \mathbb{R})$ acts via conjugation on S_{2n}. By b) J is in S_{2n}. Its stabilizer is $\mathrm{Stab}_{\mathrm{Sp}(2n,\mathbb{R})}(J) = \mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{SO}(2n, \mathbb{R}) = \mathrm{Sp}(2n, \mathbb{R}) \cap O(2n, \mathbb{R})$. Indeed

$$\begin{aligned} g \in \mathrm{Stab}_{\mathrm{Sp}(2n,\mathbb{R})}(J) &\iff gJg^{-1} = J \\ &\overset{\omega \text{ non-deg.}}{\iff} \omega(gJg^{-1}x, y) = \omega(Jx, y) \quad \forall x, y \in \mathbb{R}^{2n} \\ &\iff \omega(Jg^{-1}x, g^{-1}y) = \omega(Jx, y) \quad \forall x, y \in \mathbb{R}^{2n} \\ &\overset{\omega(J\cdot,\cdot)=\langle\cdot,\cdot\rangle}{\iff} \langle g^{-1}x, g^{-1}y\rangle = \langle x, y\rangle \quad \forall x, y \in \mathbb{R}^{2n} \\ &\iff g \in \mathrm{Sp}(2n, \mathbb{R}) \cap O(2n, \mathbb{R}). \end{aligned}$$

Using symplectic linear algebra one can show that M is ω-compatible if and only if there is a symplectic basis for $\mathbb{R}^{2n}$ of the form

$$e'_1, \ldots, e'_n, f'_1 = Me'_1, \ldots, f'_n = Me'_n,$$

i.e. $\omega(e'_i, e'_j) = 0, \omega(f'_i, f'_j) = 0, \omega(e'_i, f'_j) = \delta_{ij}$ (see [**?**, Ex. 3, p. 73]). Now define

$$\begin{aligned} g : \mathbb{R}^{2n} &\to \mathbb{R}^{2n}, \\ e_i &\mapsto e'_i \\ f_j &\mapsto f'_j \end{aligned}$$

by linear extension where $\{e_i, f_j = Je_j\}$ denotes the standard symplectic basis of $\mathbb{R}^{2n}$. Since both $\{e_i, f_j\}$ and $\{e'_i, f'_j\}$ are symplectic bases the map g is a symplectomorphism of $(\mathbb{R}^{2n}, \omega)$ onto itself, i.e. $g \in \mathrm{Sp}(2n, \mathbb{R})$. Further

$$Mg(e_i) = Me'_i = f'_i = g(f_i) = gJ(e_i)$$

and

$$Mg(f_j) = Mf'_j = -e'_j = -g(e_j) = g(Jf_j),$$

so that $Mg = gJ$, or equivalently $M = gJg^{-1}$. That shows that the action is transitive.

Therefore

$$\mathrm{Sp}(2n, \mathbb{R}) / \mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{SO}(2n, \mathbb{R}) \cong S_{2n}.$$