Exercise Sheet 3

Exercise 1. Consider $G = \mathrm{SO}(1,n)^\circ$ with the involutive Lie group automorphism

$$\sigma : G \to G, g \mapsto J_n g J_n$$

where

$$J_n = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \in \mathrm{SO}(1,n).$$

Further let

$$K = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(n) \end{pmatrix} \cong \mathrm{SO}(n).$$

It can be shown that (G, K, σ) is a Riemannian symmetric pair and that G/K is isometric to $\mathbb{H}^n$.

(a) Show that $\Theta = d\sigma : \mathfrak{g} \to \mathfrak{g}$ takes the form

$$\Theta(X) = \begin{pmatrix} 0 & -x^t \\ -x & D \end{pmatrix}$$

for all

$$X = \begin{pmatrix} 0 & x^t \\ x & D \end{pmatrix} \in \mathfrak{g} = \mathfrak{so}(1,n).$$

Deduce that

$$\mathfrak{p} = E_{-1}(\Theta) = \left\{ \begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix} : x \in \mathbb{R}^n \right\},$$
$$\mathfrak{k} = E_1(\Theta) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} : D \in \mathfrak{so}(n) \right\} \cong \mathfrak{so}(n).$$

(b) Let $\pi : G \to G/K$ denote the usual quotient map and set $\overline{X} := d_e\pi(X) \in T_o(G/K)$ for all $X \in \mathfrak{g}$. Further let $\langle X, Y \rangle := \frac{1}{2}\mathrm{tr}(XY)$ for all $X, Y \in \mathfrak{p}$.

Show that

$$R_o(\overline{X}, \overline{Y})\overline{Z} = \langle X, Z \rangle \overline{Y} - \langle Y, Z \rangle \overline{X}$$

for all $X, Y, Z \in \mathfrak{p}$. Deduce that G/K has constant sectional curvature -1.

<u>Hint:</u> You may use the following formula without proof:

The Riemann curvature tensor at $o \in M = G/K$ is given by

$$R_o(\overline{X}, \overline{Y})\overline{Z} = -\overline{[[X, Y], Z]}$$

for all $\overline{X}, \overline{Y}, \overline{Z} \in T_oM$.

(c) Compute that

$$\exp\left(t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

for all $t \in \mathbb{R}$.

Exercise 2 (Closed adjoint subgroups of $\mathrm{SL}_n(\mathbb{R})$ and their symmetric spaces)**.** Consider the Riemannian symmetric pair (G, K, σ) where $G = \mathrm{SL}_n(\mathbb{R})$, $K = \mathrm{SO}(n, \mathbb{R})$ and $\sigma : \mathrm{SL}_n(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R}), g \mapsto {}^t(g^{-1})$. Further let $H \leq G$ be a closed, connected subgroup that is adjoint, i.e. it is closed under transposition $h \mapsto {}^t h$.

(a) Show that $(H, H \cap K, \sigma|_H)$ is again a Riemannian symmetric pair.

(b) Show that $i : H \hookrightarrow G$ descends to a smooth embedding $\varphi : H/H \cap K \hookrightarrow G/K$ such that its image is a totally geodesic submanifold of G/K.

Exercise 3 (The symplectic group $\mathrm{Sp}(2n, \mathbb{R})$)**.** Let $H = \mathrm{Sp}(2n, \mathbb{R}) = \{g \in \mathrm{GL}_{2n}(\mathbb{R}) : g^t J g = J\}$ be the symplectic group, where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

(a) Show that $\mathrm{Sp}(2n, \mathbb{R}) \leq \mathrm{SL}(2n, \mathbb{R}) =: G$ is a closed connected *adjoint* subgroup of G.

What can we deduce from exercise 2 about $(H, H \cap K, \sigma|_H)$?

(b) Denote by $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ the standard symplectic form given by $\omega(x, y) = x^t J y$.

Show that $B : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}, (x, y) \mapsto \omega(Jx, y)$ is a symmetric positive definite bilinear form.

(c) An endomorphism $M \in \mathrm{End}(\mathbb{R}^{2n})$ is called a complex structure if $M^2 = -\mathrm{Id}$. We say that M is ω-compatible if $(x, y) \mapsto \omega(Mx, y)$ is a symmetric positive definite bilinear form. Denote the set of all ω-compatible complex structures by S_{2n}.

Show that $H = \mathrm{Sp}(2n, \mathbb{R})$ acts on S_{2n} via conjugation and deduce that there is a bijection $S_{2n} \cong H/H \cap K$.