Exercise Sheet 4

Part of the first exercise is to complete the proof of Theorem II.29. The numbering matches that of the lectures — be warned, it might differ slightly in the notes. Exercises 2 and 3 rely on material that you will see on Wednesday 06 November, or the week after.

Exercise 1 (Theorem II.29 - Decomposition of OSLA)**.** Let $(\mathfrak{g}, \theta)$ be an effective orthogonal symmetric Lie-algebra. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$. We decomposed $\mathfrak{e} = \mathfrak{e}_0 \oplus \mathfrak{e}_+ \oplus \mathfrak{e}_-$ and defined $\mathfrak{u}_+ = [\mathfrak{e}_+, \mathfrak{e}_+]$ and $\mathfrak{u}_- = [\mathfrak{e}_-, \mathfrak{e}_-]$. $\mathfrak{u}_0$ is defined to be the orthogonal complement of $\mathfrak{u}_+ \oplus \mathfrak{u}_-$ in $\mathfrak{u}$.

(a) Find an OSLA $(\mathfrak{g}, \theta)$, such that $\mathfrak{e}_0 = 0$, but $\mathfrak{u}_0 \neq 0$.

(b) Complete the proof of Lemma II.31 (3): that is, show that $[\mathfrak{u}_\mp, \mathfrak{e}_0] = [\mathfrak{u}_\mp, \mathfrak{e}_\pm] = (0)$.

Hint: use Lemma II.30.

(c) Prove Corollary II.32: show that $\mathfrak{u}_\varepsilon \oplus \mathfrak{e}_\varepsilon$, $\varepsilon \in \{-1, 0, +1\}$, are pairwise orthogonal ideals in $\mathfrak{g}$ (with respect to $B_\mathfrak{g}$).

(d) Let $\mathfrak{n} \lhd \mathfrak{g}$ be an ideal of a Lie-algebra $\mathfrak{g}$. Prove that $B_\mathfrak{n} = B_\mathfrak{g}|_{\mathfrak{n}\times\mathfrak{n}}$.

(e) Find an example of a subalgebra $\mathfrak{n} \subset \mathfrak{g}$, such that $B_\mathfrak{n} \neq B_\mathfrak{g}|_{\mathfrak{n}\times\mathfrak{n}}$.

(f) Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ a direct sum of two ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Further let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be subalgebras of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Show that $\mathfrak{k}_1 + \mathfrak{k}_2$ is compactly embedded in $\mathfrak{g}$ if and only if $\mathfrak{k}_1$ and $\mathfrak{k}_2$ is compactly embedded in $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

This implies that $\mathfrak{u}_0, \mathfrak{u}_-, \mathfrak{u}_+$ are compactly embedded in $\mathfrak{g}_0, \mathfrak{g}_-$ and $\mathfrak{g}_+$.

Hint: For connected G and $K < G$, there is an isomorphism

$$K/(K \cap Z(G)) \cong \mathrm{Ad}_G(K)$$

(compare Ex Sheet 2, exercise 5(b)). Use $\mathrm{Lie}(\mathrm{Ad}_G(K)) = \mathrm{ad}_{\mathrm{Lie}(G)}(\mathrm{Lie}(K))$.

Exercise 2 (Theorem II.34 - Decomposition of simply connected RSS)**.** (a) Let $H, N \lhd G$ be two normal subgroups. Show that $[N, H] \subset N \cap H$.

(b) Let $H, N < G$ be connected subgroups. Show that $[N, H]$ is a connected subgroup of G.

(c) Let M be a simply connected Riemannian symmetric space. Then $\mathfrak{g} = \mathrm{Lie}(\mathrm{Iso}(M)^\circ) = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$. We get corresponding Lie-subgroups G_0, G_+, G_- and their universal covers $\tilde{G}_0, \tilde{G}_+, \tilde{G}_-$. Let K_0, K_+, K_- be the Lie-subgroups associated to $\mathfrak{k}_0, \mathfrak{k}_+, \mathfrak{k}_-$, which come from the Cartan-decomposition of $\mathfrak{g}_0, \mathfrak{g}_+, \mathfrak{g}_-$.

Show that $(\tilde{G}_0, K_0), (\tilde{G}_+, K_+)$ and $(\tilde{G}_-, K_-)$ are Riemannian symmetric pairs.

Exercise 3 (Complexification and Killing form)**.** Let $\mathfrak{l}_0$ be a Lie algebra over $\mathbb{R}$ and let $\mathfrak{l}$ be the complexification of $\mathfrak{l}_0$. Let K_0, K and $K^{\mathbb{R}}$ denote the Killing forms of the Lie algebras $\mathfrak{l}, \mathfrak{l}_0$ and $\mathfrak{l}^{\mathbb{R}}$, respectively. Show that:

(a) $K_0(X, Y) = K(X, Y)$ for all $X, Y \in \mathfrak{l}_0$;

(b) $K^{\mathbb{R}}(X, Y) = 2 \cdot \Re(K(X, Y))$ for all $X, Y \in \mathfrak{l}^{\mathbb{R}}$.

Exercise 4 (Exceptional isogeny of $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SO}(3, 1)$)**.** Let $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and consider the vector space

$$V = \{x \in M_2(\mathbb{C}) \mid x^* = JxJ^{-1}\}$$

(where x^* denotes conjugate-transpose), endowed with the $\mathbb{R}$-bilinear $\mathbb{R}$-valued bilinear form

$$\langle x, y \rangle := \mathrm{Re}(\mathrm{Tr}(x\overline{y})))$$

(where $\overline{y}$ denotes componentwise conjugation).

Consider the $G = \mathrm{SL}(2, \mathbb{C})$ action on V given by $g \cdot x := gx\overline{g}^{-1}$, and use this to obtain an isogeny $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(3, 1)$ (that is, show this gives a surjective homomorphism with finite kernel).