# Exercise Sheet 5

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**Exercise 1** (Duality of $\mathbb{S}^n$ and $\mathbb{H}^n$)**.** Show that the symmetric spaces $\mathbb{S}^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathbb{H}^n \cong \mathrm{SO}(1,n)^\circ/\mathrm{SO}(n)$ are dual to each other.

**Solution.** Recall that we have seen in the lecture that

$$(\mathrm{SO}(n+1), \mathrm{SO}(n), \sigma) \text{ and } (\mathrm{SO}(1,n)^\circ, \mathrm{SO}(n), \sigma)$$

are Riemannian symmetric pairs, where $\sigma(g) := I_{1,n} g I_{1,n}$ in both cases. Further we have seen that the associated symmetric spaces $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathrm{SO}(1,n)^\circ/\mathrm{SO}(n)$ are isometric to the $n$-sphere $\mathbb{S}^n$ and (real) hyperbolic $n$-space $\mathbb{H}^n$ respectivley. (These are Example (3) after Corollary II.18 and exercise 1 of Exercise Sheet 3, respectively).

These have $(\mathfrak{so}(n+1), \zeta)$ and $(\mathfrak{so}(1,n), \zeta)$ as orthogonal symmetric Lie algebras, respectively, where $\zeta(X) = d\sigma(X) = I_{1,n} X I_{1,n}$ in both cases.

We have also seen in the lecture that the orthogonal symmetric Lie algebras $(\mathfrak{so}(p+q), \zeta_{p,q})$ and $(\mathfrak{so}(p,q), \zeta_{p,q})$ are dual to each other for all $p, q \geq 1$ where $\zeta_{p,q}(X) = I_{p,q} X I_{p,q}$ in both cases. Thus for $p = 1, q = n$ we obtain the assertion.

**Exercise 2** (CAT(0) normed spaces)**.** The goal of this exercise is to show that a normed vector space is CAT(0) if and only if this norm is induced by an inner product.

(a) Let $X$ be a CAT(0) space, and let $\sigma, \tau : [0,1] \to X$ be two geodesics. Show that the function $f : t \mapsto d(\sigma(t), \tau(t))$ is convex.

Recall that $f$ is convex if for any $0 \leq a < b \leq 1$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a) + f(b)}{2}$$

*Hint: consider the point $\sigma(\frac{a+b}{2})$ and the midpoint of $\sigma(a)$ and $\tau(b)$ in a suitable comparison triangle.*

(b) Conclude that a CAT(0) space $X$ is contractible.

*Hint: Use the fact that $X$ is uniquely geodesic.*

(c) Show that if $p \in X$ and $\sigma : [0,1] \to X$ is a geodesic from $x \to y$, then

$$d(p, \sigma(t))^2 \leq (1-t) d(p,x)^2 + t d(p,y)^2 - t(1-t) d(x,y)^2$$

for all $t \in [0,1]$.

(d) Deduce the *midpoint inequality*: if $p, x, y \in X$ and $z$ is a midpoint between $x$ and $y$ (that is, the point halfway along the geodesic segment joining $x$ and $y$), then

$$d(p,z)^2 \le \tfrac{1}{2}(d(p,x)^2 + d(p,y)^2) - \tfrac{1}{4}d(x,y)^2$$

(e) Suppose now $X, ||\cdot||$ is a normed real vector space, and that it is CAT(0) with respect to the induced metric. Show that it satisfies the *parallelogram law*: for any $x, y \in X$

$$||x+y||^2 + ||x-y||^2 = 2\left(||x||^2 + ||y||^2\right)$$

(f) Show that a norm on a real vector space $X$ arises from an inner product if and only if the norm satisfies the parallelogram law. In this case, the inner product is recovered by setting

$$\langle x, y\rangle := \tfrac{1}{4}\left(||x+y||^2 - ||x-y||^2\right)$$

(g) Conclude that a normed real vector space $(X, ||\cdot||)$ is a CAT(0) space if and only if the norm is induced by an inner product.

**Solution.** (a) This is Proposition III.1 (3) in the lectures.

(b) Fix some base point $o \in X$. Since $X$ is uniquely geodesic, for any $x \in X$ we can define the geodesic $\sigma_x : [0,1] \to X$ from $o$ to $x$ (notice it at constant, but not necessarily unit speed). Then the map

$$X \times [0,1] \to X : (x,t) \mapsto \sigma_x(t)$$

is a homotopy from the identity on $X$ to the constant map $x \mapsto o$.

To see that it is continuous for any $(x,t), (x',t') \in X \times [0,1]$ we calculate

$$\begin{aligned} d(\sigma_x(t), \sigma_{x'}(t')) &\le d(\sigma_x(t), \sigma_{x'}(t)) + d(\sigma_{x'}(t), \sigma_{x'}(t')) \\ &\le t d(x,x') + |t-t'| d(o,x') \end{aligned}$$

where we have used part (a) and the constant speed parameterisation of $\sigma_{x'}$.

(c) Consider the comparison triangle $\overline{x} = 0, \overline{y}, \overline{p}$. In particular by the CAT(0) property we must have that $d(\sigma(t), p) \le ||t\overline{y} - \overline{p}||$ (here $||\cdot||$ denotes the Euclidean norm). Note furthermore that

$$||t\overline{y} - \overline{p}||^2 - t||\overline{y} - \overline{p}||^2 = (t^2 - t)||\overline{y}||^2 + (1-t)||\overline{p}||^2$$

(this is a similar verification using inner products) and the result follows.

(d) Substitute $t = \frac{1}{2}$ in (c).

(e) Apply (d) with $p = 0$ to see that

$$||(x+y)/2||^2 \le \frac{||x||^2 + ||y||^2}{2} - \frac{||x-y||^2}{4} \tag{1}$$

Now apply (d) with $(x+y, x-y, 0)$ to see that

$$||x||^2 \leq \frac{||x+y||^2 + ||x-y||^2}{2} - \frac{||2y||^2}{4} \tag{2}$$

Now (1) gives us the $\leq$ in the parallelogram law, and (2) is the $\geq$.

(f) This is a standard verification, for the details see Proposition 4.4 in Brisdon and Haefliger's *Metric Spaces of Non-Positive Curvature.*

(g) By the above we know that if a CAT(0) space has a real norm then it must be induced by an inner product. The converse is clear (for any $x, y, z$ translate so that $x = 0$, and then work in the plane spanned by $y$ and $z$).

Remark: It can be shown that (c) and (d) are in fact equivalent to the CAT(0) property, we won't need this.

**Exercise 3.** Show that a graph is CAT(0) if and only if it is a tree (Here the metric is given by identifying each edge with the interval $[0, 1]$).

**Solution.** That a tree is CAT(0) is easy (draw any — all the triangles are tripods, and so distances between comparison points will always be 0). Conversely, take any graph that isn't a tree. It will have some cycle, take a minimal one, this can be used to disprove the CAT(0) inequality.

**Exercise 4.** Let $G$ be a topological group, $\mathcal{H}$ a real Hilbert space, and $\alpha : G \to \mathrm{Isom}(\mathcal{H})$ an action by isometries such that for any $x \in \mathcal{H}$ the map

$$G \to \mathcal{H}, \quad g \mapsto \alpha(g)x$$

is continuous.

It is a fact (the *Mazur-Ulam Theorem*) that such an action is by affine isometries. That is, we have two functions $\pi : G \to O(\mathcal{H})$ and $b : G \to \mathcal{H}$ such that for any $g \in G$ and $x \in \mathcal{H}$,

$$\alpha(g)x = \pi(g)x + b(g)$$

(a) Show that $b$ satisfies the *cocycle condition*

$$b(gh) = b(g) + \pi(g)b(h)$$

for all $g, h \in G$.

(b) Show that $\alpha$ has a fixed point $-\xi$ if and only if

$$b(g) = \pi(g)\xi - \xi$$

for all $g \in G$.

(c) Show that the following are equivalent:

(i) $\alpha$ has a fixed point;

(ii) $\alpha$ has a bounded orbit;

(iii) every orbit of $\alpha$ is bounded;

(iv) $b$ is bounded.

*Hint: Hilbert spaces are CAT(0).*

**Solution.** (a) We must have that $\pi(gh)x + b(gh) = \alpha(gh)x = \alpha(g)\alpha(h)x$ for all $x \in \mathcal{H}$. We calculate the right hand side:

$$\begin{aligned}\alpha(g)\alpha(h) &= \alpha(g)(\pi(h)x + b(h)) \\ &= \pi(g)(\pi(h)x + b(h)) + b(g) \\ &= \pi(g)\pi(h)x + (\pi(g)b(h) + b(h))\end{aligned}$$

and hence the cocycle condition must hold.

(b) Observe that

$$\xi = \alpha(g)(-\xi) \quad \forall g \in G \iff \xi = -\pi(g)\xi + b(g) \quad \forall g \in G \iff b(g) = \pi(g)\xi - \xi \quad \forall g \in G$$

as required.

(c) (ii), (iii), and (iv) are equivalent since for every $g \in G$ and $x \in \mathcal{H}$,

$$\alpha(g)x = \pi(g)x + b(g) \quad \text{and} \quad ||\pi(g)x|| = ||x||$$

($\pi$ is orthogonal). Clearly (i) implies (ii), so it just remains to prove the converse.

Suppose that we have a bounded orbit $S = \alpha(G) \cdot x_0$. By exercise 2 we know that $\mathcal{H}$ is CAT(0) and hence by Proposition III.1 (1) in the lectures, there exists a unique $x_S \in X$ such that $S \subset \overline{B}(x_S, r_S)$, where

$$r_S := \inf\{r > 0 \mid S \subset \overline{B}(x, r) \text{ for some } x \in X\}.$$

For any $g \in G$, $\alpha(g)S = S$ and hence by unqiueness of $x_S$ we must have that $\alpha(g)x_S = x_S$, and we have (i).