Exercise Sheet 5

Exercise 1 (Duality of $\mathbb{S}^n$ and $\mathbb{H}^n$)**.** Show that the symmetric spaces $\mathbb{S}^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathbb{H}^n \cong \mathrm{SO}(1,n)^\circ/\mathrm{SO}(n)$ are dual to each other.

Exercise 2 (CAT(0) normed spaces)**.** The goal of this exercise is to show that a normed vector space is CAT(0) if and only if this norm is induced by an inner product.

(a) Let X be a CAT(0) space, and let $\sigma, \tau : [0,1] \to X$ be two geodesics. Show that the function $f : t \mapsto d(\sigma(t), \tau(t))$ is convex.

Recall that f is convex if for any $0 \leq a < b \leq 1$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}$$

Hint: consider the point $\sigma(\frac{a+b}{2})$ and the midpoint of $\sigma(a)$ and $\tau(b)$ in a suitable comparison triangle.

(b) Conclude that a CAT(0) space X is contractible.

Hint: Use the fact that X is uniquely geodesic.

(c) Show that if $p \in X$ and $\sigma : [0,1] \to X$ is a geodesic from $x \to y$, then

$$d(p,\sigma(t))^2 \leq (1-t)d(p,x)^2 + td(p,y)^2 - t(1-t)d(x,y)^2$$

for all $t \in [0,1]$.

(d) Deduce the *midpoint inequality*: if $p, x, y \in X$ and z is a midpoint between x and y (that is, the point halfway along the geodesic segment joining x and y), then

$$d(p,z)^2 \leq \tfrac{1}{2}(d(p,x)^2 + d(p,y)^2) - \tfrac{1}{4}d(x,y)^2$$

(e) Suppose now $X, ||\cdot||$ is a normed real vector space, and that it is CAT(0) with respect to the induced metric. Show that it satisfies the *parallelogram law*: for any $x, y \in X$

$$||x+y||^2 + ||x-y||^2 = 2\left(||x||^2 + ||y||^2\right)$$

(f) Show that a norm on a real vector space X arises from an inner product if and only if the norm satisfies the parallelogram law. In this case, the inner product is recovered by setting

$$\langle x, y\rangle := \tfrac{1}{4}\left(||x+y||^2 - ||x-y||^2\right)$$

(g) Conclude that a normed real vector space $(X, ||\cdot||)$ is a CAT(0) space if and only if the norm is induced by an inner product.

Remark: It can be shown that (c) and (d) are in fact equivalent to the CAT(0) property, we won't need this.

Exercise 3. Show that a graph is CAT(0) if and only if it is a tree (Here the metric is given by identifying each edge with the interval $[0, 1]$).

Exercise 4. Let G be a topological group, $\mathcal{H}$ a real Hilbert space, and $\alpha : G \to \mathrm{Isom}(\mathcal{H})$ an action by isometries such that for any $x \in \mathcal{H}$ the map

$$G \to \mathcal{H}, \quad g \mapsto \alpha(g)x$$

is continuous.

It is a fact (the *Mazur-Ulam Theorem*) that such an action is by affine isometries. That is, we have two functions $\pi : G \to O(\mathcal{H})$ and $b : G \to \mathcal{H}$ such that for any $g \in G$ and $x \in \mathcal{H}$,

$$\alpha(g)x = \pi(g)x + b(g)$$

(a) Show that b satisfies the *cocycle condition*

$$b(gh) = b(g) + \pi(g)b(h)$$

for all $g, h \in G$.

(b) Show that α has a fixed point $-\xi$ if and only if

$$b(g) = \pi(g)\xi - \xi$$

for all $g \in G$.

(c) Show that the following are equivalent:

(i) α has a fixed point;

(ii) α has a bounded orbit;

(iii) every orbit of α is bounded;

(iv) b is bounded.

Hint: Hilbert spaces are CAT(0).