Exercise Sheet 6

Exercise 1 (Maximal abelian subspaces and regular elements in $\mathfrak{sl}(n,\mathbb{R})$)**.** Let $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$. A Cartan decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{p} = \{X \in \mathfrak{sl}(n,\mathbb{R}) : X = X^t\}$ and $\mathfrak{k} = \{X \in \mathfrak{sl}(n,\mathbb{R}) : X = -X^t\}$. We have seen in the lecture that

$$\mathfrak{a} = \left\{ \operatorname{diag}(t_1, \ldots, t_n) : t_j \in \mathbb{R}, \sum_{j=1}^n t_j = 0 \right\}.$$

is a maximal Abelian subspace of $\mathfrak{p}$.

(a) Prove (without appealing to the general theorem) that any maximal abelian subspace of $\mathfrak{p}$ is of the form $S\mathfrak{a}S^{-1}$ where $S \in SO(n)$.

(b) Show that $X \in \mathfrak{p}$ is a regular element if and only if all of its eigenvalues are distinct.

Solution. (a) Let $\mathfrak{a}'$ be a maximal abelian subspace of $\mathfrak{p}$ with basis $\{Y_1, \ldots, Y_r\}$. All of the Y_i commute pairwise, whence there is an element $S \in O(n)$ that diagonalises all of them simultaneously, that is $SY_iS^{-1} = D_i$ for every $i = 1, \ldots, r$ where D_i is some traceless diagonal matrix. Because we are free to multiply S with $\operatorname{diag}(-1, 1, \ldots, 1)$ we may assume that $S \in \mathrm{SO}(n)$. It follows, that $S\mathfrak{a}'S^{-1} \subseteq \mathfrak{a}$ and due to maximality $S\mathfrak{a}'S^{-1} = \mathfrak{a}$.

(b) Let $C_\mathfrak{g}(X)$ be the centraliser of X in $\mathfrak{g}$. Let $X \in \mathfrak{p}$ be a regular element, so $C_\mathfrak{g}(X) \cap \mathfrak{p}$ is maximal abelian. By part (b) there is $S \in \mathrm{SO}(n)$ such that

$$\mathfrak{a} = S(C_\mathfrak{g}(X) \cap \mathfrak{p})S^{-1} = C_\mathfrak{g}(SXS^{-1}) \cap \mathfrak{p}. \tag{1}$$

We have used here that $K = \mathrm{SO}(n)$ acts via the adjoint representation $\mathrm{Ad}(S)X = SXS^{-1}$, which is by Lie algebra automorphisms preserving the Cartan decomposition. Then $SXS^{-1} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) =: D$ is a diagonal matrix and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of X. Let $P_{ij} \in \mathfrak{p}$ denote the $n \times n$-permutation-matrix that permutes the canonical basis vectors $e_i \leftrightarrow e_j$ for all $i \neq j$ and fixes the rest. Then

$$[D, P_{ij}](e_k) = DP_{ij}e_k - P_{ij}De_k = 0$$

for every $k \neq i, j$,

$$[D, P_{ij}](e_i) = DP_{ij}e_i - P_{ij}De_i = (\lambda_j - \lambda_i)e_j$$

and

$$[D, P_{ij}](e_j) = DP_{ij}e_j - P_{ij}De_i = (\lambda_i - \lambda_j)e_j.$$

Thus $P_{ij} \in C_{\mathfrak{g}}(D) \cap \mathfrak{p}$ if $\lambda_i = \lambda_j$. However, $P_{ij} \notin \mathfrak{a}$ which contradicts (1).

Conversely, let $X \in \mathfrak{p}$ have distinct eigenvalues. By a theorem of the lecture we know that X is contained in a maximal abelian subspace $\mathfrak{a}'$ and there is $S \in \mathrm{SO}(n)$ such that $S\mathfrak{a}'S^{-1} = \mathfrak{a}$. Note that $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = SXS^{-1}$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of X. Obviously, $C_{\mathfrak{g}}(D) \cap \mathfrak{p} \supseteq \mathfrak{a}$. Now, let $Y \in C_{\mathfrak{g}}(D) \cap \mathfrak{p}$. For all $i, j = 1, \ldots, n$ it holds

$$0 = [Y, D]_{ij} = y_{ij}(\lambda_j - \lambda_i),$$

thus we obtain $y_{ij}\lambda_i = y_{ij}\lambda_j$ for all $i, j = 1, \ldots, n$. Since the eigenvalues of X are distinct that implies that $y_{ij} = 0$ for $i \neq j$, whence $C_{\mathfrak{g}}(D) \cap \mathfrak{p} \subseteq \mathfrak{a}$.

Exercise 2 (Maximal abelian subspaces and regular elements in $\mathfrak{sp}(2n, \mathbb{R})$)**.** Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$. Recall that a Cartan decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A = A^t, B = B^t \right\}$$

and

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A = -A^t, B = B^t \right\}.$$

(a) Define

$$\mathfrak{a} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} : A = \mathrm{diag}(t_1, \ldots, t_n). \right\}.$$

Prove that A is a maximal abelian subspace of $\mathfrak{p}$.

(b) Show that $X \in \mathfrak{p}$ is a regular element if and only if all of its eigenvalues are distinct and non-zero.

Solution. (a) It is immediate to check that $\mathfrak{a}$ is abelian. It remains to show that it is maximal abelian. Let $\mathfrak{a}' \supseteq \mathfrak{a}$ be an abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}$. Let $Y = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathfrak{a}'$. Then for every $X = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \in \mathfrak{a}$ we calculate

$$0 = [Y, X] = \begin{pmatrix} AD - DA & -BD - DB \\ BD + DB & AD - DA \end{pmatrix} = \begin{pmatrix} [A, D] & -BD - DB \\ BD + DB & [A, D] \end{pmatrix}.$$

=As in 1a) it follows that A is diagonal. Furthermore, $BD + DB = 0$ is equivalent to

$$b_{ij}(\lambda_i + \lambda_j) = 0 \qquad \forall i, j = 1, \ldots, n$$

where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ so that $B = 0$ for an appropriate choice of $\lambda_1, \ldots, \lambda_n$. That implies that $Y \in \mathfrak{a}$, whence $\mathfrak{a}' = \mathfrak{a}$, and $\mathfrak{a}$ is indeed maximal.

(b) The proof is essentially the same as for $\mathfrak{sl}(n, \mathbb{R})$. Suppose $X \in \mathfrak{p}$ is regular, that is $C_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is maximal abelian. Then there is some $k \in K := \mathrm{SO}(2n, \mathbb{R}) \cap \mathrm{Sp}(2n, \mathbb{R})$ such that

$$k(C_{\mathfrak{g}}(X) \cap \mathfrak{p})k^{-1} = C_{\mathfrak{g}}(kXk^{-1}) \cap \mathfrak{p} = \mathfrak{a}.$$

Write

$$kXk^{-1} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}$$

with $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Note that $\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n$ are the eigenvalues of X. It is easy to verify, that if $\lambda_i = \lambda_j$ for $i \neq j$ then

$$\begin{pmatrix} P_{ij} & 0 \\ 0 & -P_{ij} \end{pmatrix} \in C_{\mathfrak{g}}(X) \cap \mathfrak{p}$$

but not in $\mathfrak{a}$, and if $\lambda_i = 0$ then

$$\begin{pmatrix} 0 & E_{ii} \\ E_{ii} & 0 \end{pmatrix} \in C_{\mathfrak{g}}(X) \cap \mathfrak{p}$$

but not in $\mathfrak{a}$. Both contradicts X being regular, so $\lambda_1, \dots, \lambda_n$ must be pairwise distinct and non-zero.

Conversely, let $X \in \mathfrak{p}$ have distinct and non-zero eigenvalues. By a theorem in the lectures we know that X is contained in a maximal abelian subspace $\mathfrak{a}'$ and there is $k \in K$ such that $k\mathfrak{a}'k^{-1} = \mathfrak{a}$. Note that

$$kXk^{-1} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} =: T$$

where $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and $\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n$, are the eigenvalues of X. Obviously, $C_{\mathfrak{g}}(T) \cap \mathfrak{p} \supseteq \mathfrak{a}$. Now, let $Y \in C_{\mathfrak{g}}(T) \cap \mathfrak{p}$. Then by the same computation as in (a), we obtain $y_{ij} = 0$ for all $i \neq j$, whence $C_{\mathfrak{g}}(T) \cap \mathfrak{p} \subseteq \mathfrak{a}$.

Exercise 3 (The Siegel upper half space)**.** Let

$$\mathbb{H}_n := \{Z \in \mathbb{C}^{n \times n} : Z = Z^t,\ \operatorname{Im}(Z) \text{ is positive-definite}\}.$$

Find an explicit isomorphism between $\mathbb{H}_n$ and $\operatorname{Sp}(2n, \mathbb{R})/(\operatorname{SO}(2n) \cap \operatorname{Sp}(2n, \mathbb{R}))$. Use this and the previous exercise to construct a maximal flat of $\mathbb{H}_n$.

Hint: Consider the map

$$\begin{aligned} \varphi : \operatorname{Sp}(2n, \mathbb{R}) &\to \mathbb{H}_n, \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} &\mapsto (Ai + B) \cdot (Ci + D)^{-1}. \end{aligned}$$

Solution. The space $\mathbb{H}_n$ is also called the *Siegel upper half space*. Before we solve the problem we will show that the symplectic group $G := \operatorname{Sp}(2n, \mathbb{R})$ acts transitively via *generalized Möbius transformations*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \star Z := (AZ + B)(CZ + D)^{-1}$$

on $\mathbb{H}_n$ and that $i \cdot I_n$ has stabilizer $K := \operatorname{Sp}(2n, \mathbb{R}) \cap \operatorname{SO}(2n, \mathbb{R})$.

In order to do so, let us first see that our definition makes sense, i.e. that $(CZ+D)$ is indeed invertible and $g \star Z \in \mathbb{H}_n$ for every $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R}), Z \in \mathbb{H}_n$. Put

$$P := AZ + B, \quad Q := CZ + D.$$

Recall that $g \in \mathrm{Sp}(2n,\mathbb{R})$ is equivalent to $A^tC = C^tA$, $B^tD = D^tB$ and $A^tD - C^tB = I$. Then

$$\begin{aligned} P^t\bar{Q} - Q^t\bar{P} &= (ZA^t + B^t)(C\bar{Z} + D) - (ZC^t + D^t)(A\bar{Z} + B) && (2)\\ &= ZA^tC\bar{Z} + ZA^tD + B^tC\bar{Z} + B^tD && (3)\\ &\quad - (ZC^tA\bar{Z} + ZC^tB + D^tA\bar{Z} + D^tB) && (4)\\ &= Z - \bar{Z} = 2i \operatorname{Im} Z. && (5) \end{aligned}$$

Suppose $\xi \in \mathbb{C}^n$ is in the kernel of Q. Then

$$\xi^t \operatorname{Im} Z \bar{\xi} = \tfrac{1}{2i}\left(\xi^t P^t \bar{Q}\bar{\xi} - \xi^t Q^t \bar{P}\xi\right) = 0.$$

Since $\operatorname{Im} Z$ is positive definite, that implies that $\xi = 0$. Therefore, Q is invertible.

That $g \star Z$ is symmetric is equivalent to

$$P^tQ = (Z^tA^t + B^t)(CZ + D) = (Z^tC^t + D^t)(AZ + B) = Q^tP$$

which follows again from g being symplectic.

Also

$$\begin{aligned} \operatorname{Im}(g \star Z) &= \tfrac{1}{2i}\left(PQ^{-1} - \bar{P}\bar{Q}^{-1}\right)\\ &= \tfrac{1}{2i}\left(\left(Q^{-1}\right)^t P^t - \bar{P}\bar{Q}^{-1}\right) \end{aligned}$$

is positive definite if and only if

$$\begin{aligned} Q^t \operatorname{Im}(g \star Z)\bar{Q} &= \tfrac{1}{2i}\left(Q^t\left(Q^{-1}\right)^t P^t\bar{Q} - Q^t\bar{P}\bar{Q}^{-1}\bar{Q}\right)\\ &= \tfrac{1}{2i}\left(P^t\bar{Q} - Q^t\bar{P}\right) = \operatorname{Im} Z \end{aligned}$$

is positive definite by (5).

Let us now verify that $\star$ is an action. It is immediate that $I \star Z = Z$. Let

$$g = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}, \quad h = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R}).$$

Then

$$\begin{aligned} g \star (h \star Z) &= ((A_1 A_2 Z + A_1 B_2)(C_2 Z + D_2)^{-1} + B_1) \\ &\quad \cdot ((C_1 A_2 Z + C_1 B_2)(C_2 Z + D_2)^{-1} + D_1)^{-1} \\ &= ((A_1 A_2 Z + A_1 B_2) + B_1(C_2 Z + D_2))(C_2 Z + D_2)^{-1} \\ &\cdot (C_2 Z + D_2)((C_1 A_2 Z + C_1 B_2) + D_1 C_2 Z + D_1 D_2)^{-1} \\ &= (g \cdot h) \star Z. \end{aligned}$$

That finishes the proof that the symplectic group acts via generalized Möbius transformations on the Siegel upper half space.

Now, let us see that the action is transitive. Indeed, let $Z = X + iY \in \mathbb{H}_n$. Then the matrices

$$g = \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y^{-1}} \end{pmatrix}, \quad h = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$$

are symplectic and it is readily verified that $(hg) \star iI = X + iY$.

Finally, suppose that

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$$

stabilizes iI. Then $iI = (Ai + B)(Ci + D)^{-1}$ which is equivalent to $B = -C$ and $A = D$. Because g is symplectic we obtain $I = A^t A + B^t B$ and $A^t B = B^t A$. Therefore

$$g^t g = \begin{pmatrix} A^t & -B^t \\ B^t & A^t \end{pmatrix} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} A^t A + B^t B & A^t B - B^t A \\ B^t A - A^t B & B^t B + A^t A \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

whence $g \in \mathrm{SO}(2n, \mathbb{R})$, and $\mathrm{Stab}(iI) \subseteq K$. Vice versa, let

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K.$$

From $g^t g = I$ we obtain

$$A^t A + C^t C = I = B^t B + D^t D, \qquad 0 = A^t B + C^t D (= B^t A + D^t C)$$

and because g is symplectic

$$A^t C = C^t A, \qquad B^t D = D^t B, \qquad A^t D - C^t B = I.$$

In particular,

$$I = A^t D - C^t B + iA^t B + iC^t D = A^t(D + iB) + C^t(Di - B) \tag{6}$$

and

$$I = D^t D + B^t B = (D^t + iB^t)(D - iB),$$

whence $(D+iB)^{-1} = D^t - iB^t$. Using that relation in (6) we obtain

$$D^t - iB^t = A^t + C^t(iD - B)(D^t - iB^t) = A^t + iC^t$$

so that $A = D$ and $B = -C$. That is equivalent to $g \star iI = iI$.

This shows that $\mathbb{H}_n$ is diffeomorphic to $\mathrm{Sp}(2n,\mathbb{R})/(\mathrm{Sp}(2n,\mathbb{R}) \cap \mathrm{SO}(2n,\mathbb{R}))$ via the map

$$\varphi : G/K \to \mathbb{H}_n, gK \mapsto g \star iI.$$

By exercise 2) a maximal abelian subspace of $\mathfrak{p}$ is given by

$$\mathfrak{a} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} : A = \mathrm{diag}(t_1, \ldots, t_n). \right\}.$$

Thus a maximal flat subspace of G/K is given by $\exp(\mathfrak{a})K$. Note that

$$\exp(\mathfrak{a}) = \left\{ \begin{pmatrix} \mathrm{diag}(\lambda_1, \ldots, \lambda_n) & 0 \\ 0 & \mathrm{diag}(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}) \end{pmatrix} : \lambda_1, \ldots, \lambda_n > 0 \right\}.$$

Therefore,

$$\begin{aligned} \varphi(\exp(\mathfrak{a})K) &= \{g \star iI : g \in \exp \mathfrak{a}\} \\ &= \{i\mathrm{diag}(\lambda_1^2, \ldots, \lambda_n^2) : \lambda_i > 0\} \end{aligned}$$

is a maximal flat subspace of $\mathbb{H}_n$ via the identification $\varphi : G/K \to \mathbb{H}_n$.

Exercise 4 (Irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$)**.** Let $V = \mathbb{C}[X,Y]$ be the vector space of polynomials in two variables. Let V_m denote the vector subspace of all homogeneous polynomials of degree m. This has a basis given by the monomials $X^m, X^{m-1}Y, \ldots, Y^m$. We turn this vector subspace into a module for $\mathfrak{sl}(2,\mathbb{C})$ by defining a Lie algebra homomorphism $\varphi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V_m)$ in the following way

$$\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = X\frac{\partial}{\partial Y}, \quad \varphi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = Y\frac{\partial}{\partial X}, \quad \varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

Show that this defines an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$.

Solution. Put

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$E' = \varphi(E) = X\frac{\partial}{\partial Y}, \quad F' = \varphi(F) = Y\frac{\partial}{\partial X}, \quad H' = \varphi(H) = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

One easily checks that

$$[E, F] = H, \quad [E, H] = -2E, \quad [F, H] = 2F,$$

and

$$[E', F'] = H', \quad [E', H'] = -2E', \quad [F', H'] = 2F',$$

so that φ defines a Lie algebra homomorphism.

It remains to be shown that φ is irreducible. Suppose that there is a non-trivial invariant subspace $0 \subsetneq V' \subsetneq V_m$, and let $v' \in V'$ be non-zero. Since $\deg_Y(E'v) < \deg_Y v$ for all $v \in V_m$, there is a minimal $k \in \mathbb{N}$ such that $E'^{k+1}v' = 0$. Then $0 \neq E'^k v' \in \ker E' = \mathbb{C}X^m$, i.e. $E'^k v' = \alpha X^m$ for some non-zero $\alpha \in \mathbb{C}$. Now, applying F' successively to αX^m yields the full basis $\{X^m, X^{m-1}Y, \ldots, XY^{m-1}, Y^m\}$ which is contained in V' by invariance. However, that implies that $V' = V_m$ contradicting our assumption.