Solutions to Exercise sheet 10:

1. Any continuous function f on $[0,1]$ is uniformly continuous and so for any $\epsilon > 0$, there exists N such that for any $n \geq N$, we have that $|\mu_n(f) - \int f\, dx| < \epsilon$. And so, we have that $\mu_n \to^{w*} dx$ the Lebesgue measure on $[0,1]$.

2. See solution of Q2 of HS2023 Exercise sheet 11.

3. See solution of Q3 of HS2023 Exercise sheet 11.

   This solution is not quite correct as it assumes that the compact, convex set A is *sequentially compact* which for general topological vector spaces doesn't follow from compactness. A remedy is the following:

   For a fixed $v \in A$, let $J_n := \{T_{F_k}(v) : k \geq n\}$ and so any finite intersection of the J_n is non-empty. Thus $\cap_{i=1}^N \overline{J_i} \neq \emptyset$ and so, by compactness of A, we have $\cap_{i=1}^\infty \overline{J_i} \neq \emptyset$. Let $x \in \cap_{i=1}^\infty \overline{J_i}$. We want to show that $gx = x$ for all $g \in G$.

   Fix $g \in G$ and note that the function $v \mapsto \|gv - v\|$ is continuous on A. If $\|gx - x\| = c \neq 0$, then the set $O := \{v \in A : c/2 < \|gv - v\| < 2c\}$ is open and $x \in O$. But the computation in the solution shows that $\|gT_{F_n}(v) - T_{F_n}v\| \to 0$ as $n \to \infty$ and so $\{\|gy - y\| : y \in J_n\} \cap O = \emptyset$ for n large enough which contradicts $x \in \overline{J_n}$.

   Remark: The above trick recovers a form of convergence in arbitrary compact sets and is axiomatized in the notion of *nets*. Even though it is in general false that a compact set is sequentially compact, it is true that a set is compact if and only if every net has a convergent subnet. For more info, see section 4 of https://www.mathematik.hu-berlin.de/ wendl/Sommer2017/Topologie1/nets.pdf