Solutions to Exercise sheet 11:

1. See solution of Q5 of HS2023 Exercise sheet 11.

2. See pg. 6-35 of the additional notes 'On Ameanability'.

3. If $c_0(\mathbb{N})$ were the dual of a normed space X, then Banach-Alaoglu would imply the closed unit ball B of c_0 is weak*-compact and so Krein-Milman would imply $B = \overline{\text{conv}(\text{ex}(B))}^{w*}$ but this leads to a contradiction as $\text{ex}(B) = \emptyset$. To show this final claim, let $f \in c_0$ be an extreme point of B. There exists n_1 so that $|f(n_1)| < 1$ and so there exists $x, y \in \mathbb{K}$ such that $|x|, |y| < 1$ and $f(n_1) = (x+y)/2$. Now let $f_1(n) := f(n)(1-\delta_{n=n_1}) + x\delta_{n=n_1}$ and $f_2(n) := f(n)(1-\delta_{n=n_1}) + y\delta_{n=n_1}$. We see that $f_1, f_2 \in B$ but $f = (f_1 + f_2)/2$ contradicting that f is an extreme point.

4. Any mean $v \in \mathcal{M}$ gives rise to a finitely additive probability measure μ_v on $\mathbb{N}$ by setting $\mu_v(E) = v(\chi_E)$. Now if there exists a set $E \subset \mathbb{N}$ such that $\mu_v(E), \mu_v(\mathbb{N} \setminus E) > 0$ then v is not an extreme point. To see this note that $v_1(f) := \mu_v(E)^{-1}v(\chi_E f)$ and $v_2(f) := \mu_v(\mathbb{N} \setminus E)^{-1}v(\chi_{\mathbb{N}\setminus E} f)$ are elements of $\mathcal{M}$ and that $v = \mu_v(E)v_1 + \mu_v(\mathbb{N} \setminus E)v_2$.

   Thus to be an extreme point we must have that for any $E \subset \mathbb{N}$ the set $\{\mu_v(E), \mu_v(\mathbb{N} \setminus E)\} = \{0, 1\}$. Say a mean v satisfies this property and $v = av_1 + bv_2$ for $a, b > 0$ and $a + b = 1$. If $\mu_v(E) = 1$ then $\mu_{v_1}(E), \mu_{v_2}(E) = 1$ and if $\mu_v(E) = 0$ then $\mu_{v_1}(E), \mu_{v_2}(E) = 0$ and so $\mu_{v_1}(E) = \mu_{v_2}(E)$ for all $E \subset \mathbb{N}$. So, by approximating any ℓ^∞ function f by simple functions, we conclude $v_1(f) = v_2(f)$. So $v_1 = v_2$ and v is an extreme point.

   As the set of means $\mathcal{M}$ is a closed, convex subset of the closed unit ball of $\ell^\infty(\mathbb{N})^*$, the Banaach-Alouglu theorem implies it is weak*-compact and so by the Krein-Milman theorem, we have $\mathcal{M} = \overline{\text{conv}(\text{ex}(\mathcal{M}))}^{w*}$. But, by Hahn-Banach, we can construct an element m of $\mathcal{M}$ that is 0 on all elements of ℓ^∞ with finite support (why exactly?). But now if all the extreme points don't vanish on some element with finite support, by the above characterization, they are of the form m_n where $m_n(f) = f(n)$. But now $m \notin \overline{\text{conv}(\text{ex}(\mathcal{M}))}^{w*}$ contradicting Krein-Milman.

   Remark: The extreme points of $\mathcal{M}$ are ultrafilter limits of elements in ℓ^∞ corresponding to the ultrafilters $\beta\mathbb{N}$. The ones that vanish on all functions with finite support correspond to the non-principal ultrafilters.