Exercise Sheet 12

1. Let $\delta > 0$ and $f_\delta : \mathbb{R} \to \mathbb{R}$ be defined as

$$f_\delta(x) = \begin{cases} 1 \text{ for } x \in [-1, 1] \\ -\delta^{-1}|x| + 1 + \delta^{-1} \text{ for } x \in [-1-\delta, -1] \cup [1, 1+\delta] \\ 0 \text{ for } x \notin [-1-\delta, 1+\delta] \end{cases}$$

   Compute $\hat{f}_\delta$ and show that $\lim_{\delta\to 0} \|\hat{f}_\delta\|_1 = \infty$

2. Show that there are $f \in C_{00}(\mathbb{R})$ with $\hat{f} \notin L^1(\mathbb{R})$. Proceed by contradiction. Assume that $\forall f \in C([-2, 2])$ we have $\hat{f} \in L^1(\mathbb{R})$. Apply the closed graph theorem to obtain a contradiction using exercise 1.

3. Let $f \in L^2(\mathbb{R}^n)$. For $R > 0$, show that $f\chi_{\le R} \in L^1(\mathbb{R}^n)$ and show that as $R \to \infty$

$$\widehat{f\chi_{\le R}} \to \mathcal{F}f \text{ in } L^2(\mathbb{R}^n)$$

   where $\mathcal{F}f$ is the L^2-Fourier transform.

4. For an $f_1, f_2, f_3 \in L^1(\mathbb{R}^n)$ show that $f_1 * f_2 = f_2 * f_1$ and

$$(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3)\,.$$