Solutions to Exercise sheet 12:

1.

2. First note that the mapping $f \mapsto \hat{f}$ is an injective map from $L^1(\mathbb{R})$ to $C_0(\mathbb{R})$. Indeed, if an element $f \in L^1(\mathbb{R})$ satisfies $\hat{f} = 0$ and η_ϵ be an approximation of the identity, then $f * \eta_\epsilon \in C_c^\infty(\mathbb{R})$ and so by Fourier inversion
$$\mathcal{F}^* \widehat{f * \eta_\epsilon} = f * \eta_\epsilon \,.$$
But now $\widehat{f * \eta_\epsilon} = \hat{f}\hat{\eta}_\epsilon = 0$ and $f * \eta_\epsilon \to f$ in $L^1(\mathbb{R})$ as $\epsilon \to 0$ gives that $f = 0$.

   Now, if every element $g \in C_0([-2, 2])$ is of the form $g = \hat{f}$ for some $f \in L^1(\mathbb{R})$, then the map $A : C_0([-2, 2]) \to L^1(\mathbb{R})$ sending $g \mapsto f$ is well-defined and linear. Now, if $f_n \to f$ in $L^1(L^1(\mathbb{R})$ and $\hat{f}_n \to g$ in $C_0([-2, 2])$, then as $\hat{f}_n \to \hat{f}$ by boundedness of the Fourier transform, we have that $g = f$ and, therefore, by the closed graph theorem, the map A is bounded. But now if $\|\hat{f}\|_\infty = 1$ for $\hat{f} \in C_0([-2, 2])$ then $\|f\|_{L^1} \leq C$ for some $C > 0$ but the first exercise provides a counterexample to this statement.

3. We have
$$\|f\chi_{\leq R}\|_{L^1} \leq \|f\|_{L^2}\|\chi_{\leq R}\|_{L^2} \leq \|f\|_{L^2}\sqrt{2R} < +\infty$$
and so $f\chi_{\leq R} \in L^1(\mathbb{R})$.

   Also, as $\|f\chi_{\leq R}\|_{L^2} \leq \|f\|_{L^2}\|\chi_{\leq R}\|_\infty < +\infty$, we have $f\chi_{\leq R} \in L^2(\mathbb{R})$. So, $f\chi_{\leq R} \in L^1 \cap L^2$ and so, by Plancherel's theorem, we have $\widehat{f\chi_{\leq R}} = \mathcal{F}f\chi_{\leq R} \in L^2$. Now, note that $f\chi_{\leq R} \to f$ in L^2 and so we conclude by noting
$$\|\widehat{f\chi_{\leq R}} - \mathcal{F}f\|_{L^2} = \|\mathcal{F}(f\chi_{\leq R} - f)\|_{L^2} = \|f\chi_{\leq R} - f\|_{L^2} \to 0\,.$$

4. See solutions to exercise 3 of HS2023 question sheet 13.