Solutions to Exercise sheet 13:

1. See top of pg. 82 of the script.

2. If $f \in L^1(\mathbb{R}^n)$ satisfies $f * f = f$ then, by taking the Fourier transform we obtain $\hat{f}^2 = \hat{f}$. Thus we have $\hat{f}(\xi) \in \{0, 1\}$. But $\hat{f} \in C_0(\mathbb{R}^n)$ and so, we must have $\hat{f} \equiv 0$ which implies tat $f \equiv 0$ by the solution to Q2 of exercise sheet 12.

3. Consider for $\varphi \in C_c^\infty(\mathbb{R})$ and $\epsilon > 0$ the integral

$$\int_\epsilon^\infty |x|^r \varphi'(x)\, dx = -|\epsilon|^r \varphi(\epsilon) - \int_\epsilon^\infty r|x|^{r-1}\varphi(x)\, dx\,.$$

The integration by parts makes sense as the functions considered are C^1 in the region considered. Similarly,

$$\int_{-\infty}^{-\epsilon} |x|^r \varphi'(x)\, dx = |\epsilon|^r \varphi(-\epsilon) - \int_{-\infty}^{-\epsilon} -r|x|^{r-1}\varphi(x)\, dx\,.$$

Now we have

$$\begin{aligned}\int_{\mathbb{R}} |x|^r \varphi'(x)\, dx &= \int_{-\infty}^{-\epsilon} |x|^r \varphi'(x)\, dx + \int_{-\epsilon}^{\epsilon} |x|^r \varphi'(x)\, dx + \int_\epsilon^\infty |x|^r \varphi'(x)\, dx \\ &= -\int_{\mathbb{R}} \operatorname{sgn}(x) r|x|^{r-1}\varphi(x)\, dx + \int_{-\epsilon}^{\epsilon} |x|^r \varphi'(x) - \operatorname{sgn}(x) r |x|^{r-1}\varphi(x)\, dx\,.\end{aligned}$$

The last term above can be bounded as

$$\int_{-\epsilon}^{\epsilon} |x|^r |\varphi'(x)| + r|x|^{r-1}|\varphi(x)|\, dx \le 2(\epsilon^{r+1}\|\varphi'\|_\infty + \epsilon^r \|\varphi\|_\infty) \to 0$$

as $\epsilon \to 0$.

4. All terms are well defined in the sum as $f \in W^{k,2}$ implies that $D_w^\alpha f \in L^2$ for all $|\alpha| \le k$. Since $\langle f, g\rangle \ge \langle f, g\rangle_{L^2}$, we get that $\langle f, f\rangle = 0$ implies $f = 0$. Symmetry and triangle inequality are immediate from properties of the L^2 inner product. Thus we have a well-defined inner product.

Let $C(k, n)$ be the number of multi-indices α such that $|\alpha| \le k$. Now by Cauchy-Schwartz, we have

$$\|f\|_{W^{k,2}} \le \langle f, f\rangle^{1/2} C(k, n)^{1/2}\,.$$

Also,

$$\langle f, f\rangle \le \left(\sum_{|\alpha|\le k} \|D_w^\alpha f\|_{L^2}\right)^2 \le \|f\|_{W^{k,2}}^2$$

allows us to conclude.