Exercise Sheet 1

1. Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$.

   (a) Prove: For all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $\|x\| \leq 1$ and $\|y\| \leq 1$ satisfy $\|x - y\| > \epsilon$ then $\|\frac{x+y}{2}\| < 1 - \delta$. Compute δ as a function of ϵ.

   (b) Draw a picture of this geometric property.

2. Let $\mathcal{H}$ be a Hilbert space, $x, y, z \in \mathcal{H}$, $c : \mathbb{R} \to \mathcal{H}$, $t \mapsto tx + (1-t)y$ be a parametrization of the line through $x$and y and $f(t) := \|z - c(t)\|^2$. Assuming $x \neq y$, show that f is strictly convex.

3. Let $C \subset \mathcal{H}$ be a closed, convex subset of a Hilbert space and set $d(x, C) := \inf\{\|x - y\| : y \in C\}$ for all $x \in \mathcal{H}$. Show that for each $x \in \mathcal{H}$, there is a unique point $p(x) \in C$ which satisfies $d(x, C) = d(x, p(x))$.
   Hint: Let $(x_n)_n$ be a sequence in C with $d(x, x_n) \to d(x, C)$ as $n \to \infty$. Prove by using exercise 2 that any such sequence is Cauchy. Use exercise 3 to prove that any two points x_1, x_2 which satisfy $d(x, x_1) = d(x, x_2) = d(x, C)$ are equal.

4. Let $c_{00} = \{f : \mathbb{Z} \to \mathbb{C} : f \text{ has finite support}\}$ be equipped with the norm

$$\|f\|_p = \left[\sum_{m \in \mathbb{Z}} |f(m)|^p\right]^{1/p}.$$

   Show that the norms $\|\cdot\|_{p_1}, \|\cdot\|_{p_2}$ are equivalent iff $p_1 = p_2$.

5. Let $\alpha > 1$. Show that for any $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^\alpha} < \infty$$

   is constant.

6.* One can define $\bigwedge^\alpha(X)$ for any metric space (X, d). Give a simple geometric condition on the metric space (X, d) which implies that for all $\alpha > 1$ the space $\bigwedge^\alpha(X)$ consists only of constant functions.