Exercise Sheet 2

Let $(V, \|\cdot\|)$ be a normed space.

1. Let $W \subset V$ be a subspace. For any $v \in V$ define

$$d(v, W) := \inf_{w \in W} \|v - w\| .$$

   Assume $W \subset V$ is closed and $W \neq V$. Show that for all $\epsilon > 0$ there exists $v \in V$ with $\|v\| = 1$ and $d(v, W) > 1 - \epsilon$.

2. Let $W \subset V$ be a subspace. Define $\|v + W\| := d(v, W)$ for all $v + W \in V/W$.

   (a) Show that this defines a norm on V/W iff W is closed in V.

   (b) Show that if V is Banach and W is closed in V then V/W is Banach.

   (c) Prove: If W is closed and $W \neq V$ then the canonical projection

$$\pi : V \to V/W$$

   satisfies $\|\pi\| = 1$.

3. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $S := \{v \in \mathcal{H} : \|v\| = 1\}$. Show that the unitary group $U(\mathcal{H})$ acts transitively on S. What can you say about the stabilizer of a point $v \in S$?
   Let $n \in \mathbb{N}$, $n \geq 1$ and

$$\mathcal{F}(n) := \{(v_1, ..., v_n) \in \mathcal{H}^n : v_1, ..., v_n \text{ form an orthonormal set}\} .$$

   Assume $\mathcal{F}(n) \neq \emptyset$. Show that $U(\mathcal{H})$ acts transitively on $\mathcal{F}(n)$.

4. Let $S := \{v \in V : \|v\| = 1\}$. Show that the following are equivalent:

   (a) $\dim(V) < +\infty$.

   (b) S is compact.

   Hint: use exercise 1 to prove (b) implies (a).