Exercise Sheet 3

1. Let $T\colon V \to W$ be a linear map between normed vector spaces. Assume W is finite-dimensional. Show that T is continuous if and only if $\ker(T)$ is a closed subspace.

2. Let V be a $\mathbb{R}$-vector space and $C \subset V$ a convex subset such that for all $v \in V$ there exists $\lambda > 0$ with $v \in \lambda C$. Show that
$$p(v) := \inf\{\lambda > 0 : v \in \lambda C\}$$
is a gauge function on V with
$$\{v \in V : p(v) < 1\} \subset C \subset \{v \in V : p(v) \leqslant 1\}.$$

3. Let V be a normed space, $E \subset V$ a closed subspace with $E \neq V$ and $x_0 \notin E$. Prove that there exists $f \in V^*$ with $f(x_0) \neq 0$ and $E \subset \ker(f)$.

4. Let V be a normed space. Given subsets $A \subset V$ and $B \subset V^*$, we define
$$A^{\perp} := \{f \in V^* : f|_A = 0\},$$
$$^{\perp}B := \{v \in V : f(v) = 0 \text{ for all } f \in B\}.$$
   (a) Show that $A^{\perp} \subset V^*$ and $^{\perp}B \subset V$ are closed subspaces.
   (b) Let $M \subset V$ be a vector subspace. Prove the equality $\overline{M} = {}^{\perp}(M^{\perp})$.