Exercise Sheet 5

1. Let $K \in C(X \times X, \mathbb{R})$ be symmetric and $(k_k)_{k\geq 1}$ be an orthonormal basis of $L^2(X, \mu, \mathbb{R})$ consisting of eigenfunctions of T_K. Write
$$T_K k_k = \lambda_k f_k \,.$$
Observe that $\lambda_k \in \mathbb{R}$. Show that K is positive semi-definite if and only if $\lambda_k \geq 0$ for all $k \geq 1$.

2. Let X be a set and $K : X \times X \to \mathbb{R}$ be a symmetric positive semi-definite kernel. Show that there is a $\mathbb{R}$-Hilbert space $\mathcal{H}$ and a map $\varphi : X \to \mathcal{H}$ such that
$$K(x, y) = \langle \varphi(x), \varphi(y) \rangle \,.$$
Hint: Let $\mathbb{R}^{(X)} := \{f : X \to \mathbb{R} : f \text{ has finite support}\}$ and define
$$\langle f_1, f_2 \rangle = \sum_{x,y \in X} f_1(x) f_2(y) K(x, y) \,.$$
Warning: You're not done!

3. Let $\mathcal{H}$ be a $\mathbb{C}$-Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $TT^* = T^*T$. Show that if T is compact and normal, then $\mathcal{H}$ has an orthonormal basis of eigenvectors; show that $\dim(\mathcal{H}_\lambda) < +\infty$ for $\lambda \neq 0$ and that $\forall \epsilon > 0$
$$|\{\lambda \in \mathbb{C} : |\lambda| \geq \epsilon, \mathcal{H}_\lambda \neq \{0\}\}| < +\infty \,.$$