Exercise Sheet 9

1. Show that if $(V, \|\cdot\|)$ is a normed space such that V^* is separable, then the weak topology on $B^V_{\leq 1}(0)$ is metrizable.

2. Define $\lambda : \ell^1(\mathbb{N}) \to c_0(\mathbb{N})^*$ by
$$\lambda(f)(g) = \sum_{n=0}^{\infty} f(n)g(n)\,.$$
Show that λ is a bijective isometry.

3. Let $(V, \|\cdot\|)$ be a normed space. Show that any linear form $f : V^* \to \mathbb{K}$ that is continuous with respect to the weak* topology is of the form $f(h) = h(v)$ for $h \in V^*$ for some $v \in V$.

4. For a set S, define $\mathbb{R}^{(S)}_{\geq 0} := \{\lambda : S \to \mathbb{R}_{\geq 0}, \lambda \text{ has finite support}\}$. Show that the convex hull of a subset $A \subset V$ is given by
$$\operatorname{conv}(A) = \left\{ \sum_{a\in A} \lambda(a)a : \lambda \in \mathbb{R}^{(A)}_{\geq 0}, \sum_{a\in A} \lambda(a) = 1 \right\}.$$