Solutions to Exercise sheet 9:

1. As V^* is separable, we can choose a countable, dense subset $\{\phi_i\}_{i=1}^{\infty}$ for $B_{\leq 1}^{V^*}(0)$. Now define a metric d on $B_{\leq 1}^{V}(0)$ as follows

$$d(x,y) := \sum_{i=1}^{\infty} \frac{|\phi_i(x-y)|}{i^2} \,.$$

   Now check that this is indeed a metric generating the weak topology on V.

2. Firstly, since $|\sum f(n)g(n)| \leq \|f\|_{\ell^1}\|g\|_{\ell^\infty}$, we have that λ is a well-defined bounded map. Now we check that it is injective: if $f = (f_n)_{n=1}^{\infty}$ is in the kernel of λ, then by letting $e_n = (\delta_{in})_{i=1}^{\infty} \in c_0$ we have $0 = \lambda(f)g = f_n$, and so $f \equiv 0$. To show that it's surjective, see solution of Q3 of HS2023 Exercise sheet 10.

3. See solution of Q4 of HS2023 Exercise sheet 9.

4. First we show that $\operatorname{conv}(A)$ is a convex set that contains A. Clearly $A \subset \operatorname{conv}(A)$: for $a_0 \in A$, by setting $\lambda(a) = \delta_{a=a_0}$, we see $a_0 \in \operatorname{conv}(A)$. Now for $x, y \in \operatorname{conv}(A)$ given by the functions $\lambda_x, \lambda_y \in \mathbb{R}_{\geq 0}^{(S)}$, we have for $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, $\alpha\lambda_x + \beta\lambda_y \in \mathbb{R}_{\geq 0}^{(S)}$ and this function represents the element $\alpha x + \beta y \in \operatorname{conv}(A)$ and so $\operatorname{conv}(A)$ is convex.

   To show that it is the convex hull, we need to show that for any convex set X containing A, we have $\operatorname{conv}(A) \subset X$. We do this by induction on the size of the support of $\lambda \in \mathbb{R}_{\geq 0}^{(S)}$. If the support is a single element then we are done as those simply correspond to elements of A. Say, we have proved it for all sizes of support up to n. We now need to prove it for $n+1$. To this end, let λ be a function with support size $n+1$ and let $\lambda(a_0) > 0$. Now consider the functions, $\lambda_1 := \delta_{a=a_0}$ and $\lambda_1 := \frac{\lambda - \lambda(a_0)\delta_{a=a_0}}{1-\lambda(a_0)}$ in $\mathbb{R}_{\geq 0}^{(S)}$. By definition, their supports have size $\leq n$ and so the elements x, y in $\operatorname{conv}(A)$ they correspond to also belong to X by the induction hypothesis. But now, as X is convex $\lambda(a_0)x + (1-\lambda(a_0))y \in X$ and so the element corresponding to λ must also belong to X completing the induction step.