Exercise Sheet 3 - Solutions

1. Let $T\colon V \to W$ be a linear map between normed vector spaces. Assume W is finite-dimensional. Show that T is continuous if and only if $\ker(T)$ is a closed subspace.

   Solution: The kernel can be written as $\ker(T) = T^{-1}(\{0\})$. Any normed space is Hausdorff, so the subset $\{0\} \subset W$ is closed. The continuity of T implies that $T^{-1}(\{0\})$ is closed.

   Because $\ker(T)$ is closed, exercise 2 on exercise sheet 2 proves that $V/\ker(T)$ can be equipped with the structure of a normed space such that the projection map $\pi\colon V \to V/\ker(T)$ is continuous. By the universal property of the kernel, there exists a linear map $\overline{T}\colon V/\ker(T) \to W$ such that $\overline{T}\circ\pi = T$. Since the composition of continuous functions is continuous, it suffices to prove that $\overline{T}$ is continuous. This is a linear map between two finite-dimensional normed vector spaces, so it is continuous.

2. Let V be a $\mathbb{R}$-vector space and $C \subset V$ a convex subset such that for all $v \in V$ there exists $\lambda > 0$ with $v \in \lambda C$. Show that

   $$p(v) := \inf\{\lambda > 0 : v \in \lambda C\}$$

   is a gauge function on V with

   $$\{v \in V : p(v) < 1\} \subset C \subset \{v \in V : p(v) \leqslant 1\}$$

   Solution: Let $v \in V$ and $\eta > 0$. Then for all $\lambda > 0$ we have $\eta v \in \lambda C$ if and only if $v \in (\eta\lambda)C$. Thus $p(\eta v) = \eta p(v)$.

   Let $v_1, v_2 \in V$. For all $\lambda_1, \lambda_2 > 0$ with $v_1 \in \lambda_1 C$ and $v_2 \in \lambda_2 C$, the convexity of C implies

   $$(\lambda_1 + \lambda_2)^{-1}(v_1 + v_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}\frac{v_1}{\lambda_1} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\frac{v_2}{\lambda_2} \in C.$$

   Thus $v_1 + v_2 \in (\lambda_1 + \lambda_2)C$ and so $p(v_1 + v_2) \leqslant \lambda_1 + \lambda_2$. Taking the infimum yields the desired inequality $p(v_1 + v_2) \leqslant p(v_1) + p(v_2)$.

   Consider $x \in \{v \in V : p(v) < 1\}$. Note that $0 \in C$ because for any $\lambda > 0$ we have $\lambda 0 = 0$. Thus $[0, 1/p(x)) \subset \{\lambda \geqslant 0 : \lambda x \in C\}$ because C is convex. Since $p(x) < 1$, we get $x \in C$.

   The condition $x \in C$ implies $p(x) \leqslant 1$ by definition of p.

3. Let V be a normed space, $E \subset V$ a closed subspace with $E \neq V$ and $x_0 \notin E$. Prove that there exists $f \in V^*$ with $f(x_0) \neq 0$ and $E \subset \ker(f)$.

   Solution: By exercise 2 of sheet 2, the space V/E can be given the structure of a normed space such that the projection $\pi\colon V \to V/E$ is continuous. Since $\pi(x_0) \neq 0$, Corollary II.9 implies the existence of a continuous functional $\overline{f}\colon V/E \to \mathbb{K}$ with $\overline{f}(\pi(x_0)) \neq 0$. The functional $f := \overline{f} \circ \pi$ satisfies all the requirements.

4. Let V be a normed space. Given subsets $A \subset V$ and $B \subset V^*$ we define

$$A^\perp := \{f \in V^* : f|_A = 0\}$$
$$^\perp B := \{v \in V : f(v) = 0 \text{ for all } f \in B\}.$$

(a) Shot that $A^\perp \subset V^*$ and $^\perp B \subset V$ are closed subspaces.

(b) Let $M \subset V$ be a vector subspace. Prove the equality $\overline{M} = {}^\perp(M^\perp)$.

Solution:

(a) For each $x \in V$, the space $\{x\}^\perp$ is a closed subspace because it is the kernel of the continuous functional $V^* \to \mathbb{K}, f \mapsto f(x)$. We have

$$A^\perp = \bigcap_{x \in A} \{x\}^\perp,$$

so $A^\perp$ is a closed subspace. The argument for $^\perp B$ follows a very similar line of reasoning.

(b) One can see $M \subset {}^\perp(M^\perp)$ by unwrapping the definitions. By exercise (a), the space $^\perp(M^\perp)$ is closed, so $\overline{M} \subset {}^\perp(M^\perp)$.
Let $x \in {}^\perp(M^\perp)$ and suppose $x \notin \overline{M}$. By exercise 3, there exists a continuous functional $f \colon V \to \mathbb{K}$ with $f(x) \neq 0$ and $f_{\overline{M}} = 0$. In particular, we get $f \in {}^\perp M$. Since $f(x) \neq 0$, this implies $x \notin {}^\perp(M^\perp)$. This contradicts $x \in {}^\perp(M^\perp)$, so we get $x \in \overline{M}$.