Exercise Sheet 4 - Solutions

1. Show Prop. III.3 (1) and (2).

   Solution: The point space $\{0\}$ is compact, so $0 \in \mathscr{K}(V,W)$. If $K \subset W$ is compact and $\lambda \in \mathbb{K}$, then λK is compact because it is the image of K under the continuous map $m_\lambda(w) := \lambda w$. For all $T \in \mathscr{K}(V,W)$ and $\lambda \in \mathbb{K} - \{0\}$, we have $\overline{(\lambda T)(B_{\leqslant 1}(0))} = \lambda \overline{T(B_{\leqslant 1}(0))}$ because m_λ is a homeomorphism. Thus $\lambda T \in \mathscr{K}(V,W)$. Let $T_1, T_2 \in \mathscr{K}(V,W)$. Then

   $$\overline{(T_1+T_2)(B_{\leqslant 1}(0))} \subset \overline{T_1(B_{\leqslant 1}(0)) + T_2(B_{\leqslant 1}(0))} \subset \overline{\overline{T_1(B_{\leqslant 1}(0))} + \overline{T_2(B_{\leqslant 1}(0))}}.$$

   The set $\overline{T_1(B_{\leqslant 1}(0))} + \overline{T_2(B_{\leqslant 1}(0))}$ is compact because it is the image of the compact space $\overline{T_1(B_{\leqslant 1}(0))} \times \overline{T_2(B_{\leqslant 1}(0))}$ under the addition map. In particular, it is closed in W (because W is Hausdorff), so $\overline{T_1(B_{\leqslant 1}(0))} + \overline{T_2(B_{\leqslant 1}(0))} = \overline{\overline{T_1(B_{\leqslant 1}(0))} + \overline{T_2(B_{\leqslant 1}(0))}}$. Thus $\overline{(T_1+T_2)(B_{\leqslant 1}(0))}$ is a closed subspace of a compact space hence it is compact.

   We have $\overline{(BTA)(B_{\leqslant 1}(0))} = \overline{B(\overline{(TA)(B_{\leqslant 1}(0))})}$. The space $\overline{(TA)(B_{\leqslant 1}(0))}$ is compact. Indeed, $A(B_{\leqslant 1}(0))$ is bounded because A is continuous, so it has to be compact because T is compact. This implies that $\overline{B(\overline{(TA)(B_{\leqslant 1}(0))})}$ is compact because it is the (closed) image of a compact space under a continuous mapping.

2. Let $\mathscr{H}$ be a separable Hilbert space and $\{e_i : i \geqslant 1\}$ an orthonormal basis for $\mathscr{H}$.

   (a) Prove: For all complex numbers $\lambda_i \in \mathbb{C}$, the operator

   $$T\colon \bigoplus_{i\geqslant 1} \mathbb{C}e_i \to \bigoplus_{i\geqslant 1} \mathbb{C}e_i,\ e_i \mapsto \lambda_i e_i$$

   extends to a bounded operator on $\mathscr{H}$ if and only if $\sup_i |\lambda_i| < \infty$.

   (b) Show that if T is compact, then $\lim_{i\to\infty} \lambda_i = 0$.

   Solution:

   (a) Suppose $\overline{T}\colon \mathscr{H} \to \mathscr{H}$ is a bounded operator with $\overline{T}(e_i) = \lambda_i e_i$. Then

   $$|\lambda_i| = ||\lambda_i e_i|| = ||Te_i|| \leqslant ||T||.$$

   Suppose $\sup_i |\lambda_i| < \infty$. Define

   $$\overline{T}\Big(\sum_i a_i e_i\Big) := \sum_i \lambda_i a_i e_i.$$

   for all $v = \sum_i a_i e_i \in \mathscr{H}$. This is a well-defined and bounded map $\overline{T}\colon \mathscr{H} \to \mathscr{H}$ because

   $$||\overline{T}(v)||^2 = \Big|\Big|\overline{T}\Big(\sum_i a_i e_i\Big)\Big|\Big|^2 = \sum_i |\lambda_i|^2 |a_i|^2 \leqslant (\sup_i |\lambda_i|)^2 ||v||^2$$

   for all $v \in \mathscr{H}$.

(b) Suppose T is compact. If $\lim_{i\to\infty} \lambda_i \neq 0$, then there exists a subsequence $(\lambda_{i_k})_k$ and $\epsilon > 0$ with $|\lambda_{i_k}| > \epsilon$ for all k. There can not be a subsequence of $(Te_{i_k}) = (\lambda_{i_k} e_{i_k})_k$ which is Cauchy because $||\lambda_i e_i - \lambda_j e_j|| = |\lambda_i|^2 + |\lambda_j|^2$ for all $i \neq j$. This implies that T is not compact which gives the desired contradiction.

3. Let $\mathscr{H}$ be a separable Hilbert space and $\mathscr{B}_2(\mathscr{H})$ the space of Hilbert-Schmidt operators equipped with the Hilbert-Schmidt norm. Prove that $\mathscr{B}_2(\mathscr{H})$ is a Hilbert space.

Solution: Let $\{e_i : i \geqslant 1\}$ be an orthonormal basis for $\mathscr{H}$. The parallelogram identity implies

$$\begin{aligned} ||T_1 + T_2||_2^2 + ||T_1 - T_2||_2^2 &= \sum_i ||(T_1 + T_2)e_i||^2 + ||(T_1 - T_2)e_i||^2 \\ &= \sum_i 2(||T_1 e_i||^2 + ||T_2 e_i||^2) \\ &= 2(||T_1||_2^2 + ||T_2||_2^2). \end{aligned}$$

Thus the Hilbert-Schmidt norm satisfies the parallelogram identity which means it comes from an inner product.

Let $(T_n)_n$ be a Cauchy sequence in $\mathscr{B}_2(\mathscr{H})$. This is a Cauchy sequence in $\mathscr{B}(\mathscr{H})$ because $||\cdot|| \leqslant ||\cdot||_2$. Let T be the limit of the sequence $(T_n)_n$ in $\mathscr{B}(\mathscr{H})$. The triangle inequality implies

$$\begin{aligned} \sqrt{\sum_{i=1}^{M} ||(T - T_n)e_i||^2} &\leqslant \sqrt{\sum_{i=1}^{M} ||(T - T_m)e_i||^2} + \sqrt{\sum_{i=1}^{M} ||(T_m - T_n)e_i||^2} \\ &\leqslant \sqrt{M}||T - T_m|| + ||T_m - T_n||_2 \end{aligned}$$

for all n, m and N. Let $\epsilon > 0$. Pick M with $||T_n - T_m||_2 \leqslant \epsilon$ for all $m, n \geqslant M$. For all such pairs the above inequality gives

$$\sqrt{\sum_{i=1}^{M} ||(T - T_n)e_i||^2} \leqslant \sqrt{M}||T - T_m|| + \epsilon.$$

By letting $m \to \infty$ the above inequality becomes

$$\sqrt{\sum_{i=1}^{M} ||(T - T_n)e_i||^2} \leqslant \epsilon.$$

This inequality means $||T - T_n||_2 \leqslant \epsilon$ when $M \to \infty$ for all $n \geqslant N$. In particular, the operator T is Hilbert-Schmidt and T is the limit of $(T_n)_n$ in the space of Hilbert-Schmidt operators.

4. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space such that $L^2(X, \mu)$ is separable.[1] Prove that every Hilbert-Schmidt operator on $L^2(X, \mu)$ is of the form T_K for some kernel $K \in L^2(X \times X, \mu \times \mu)$.

[1] See this mathoverflow thread for a discussion on when $L^2(X)$ is separable. More general sources of examples are Radon measures on second-countable LCH spaces, e.g. smooth measures on manifolds.

Solution: Let $\{e_i\}$ be an orthonormal basis for $L^2(X,\mu)$. Write e_i^* for the dual functional

$$e_i^*\colon f \in L^2(X,\mu) \mapsto \int_X f\overline{e_i}d\mu \in \mathbb{K}.$$

Define the Hilbert-Schmidt operators $(e_j^* \otimes e_i)(f) := e_j^*(f)e_i$.

Let $T\colon L^2(X,\mu) \to L^2(X,\mu)$ be a Hilbert-Schmidt operator, then

$$\left|\left|T - \sum_{i,j=1}^{N} \langle Te_i, e_j\rangle e_j^* \otimes e_i\right|\right|_2^2 \leqslant \sum_{\substack{i\geqslant 1\\ j\geqslant N+1}} |\langle Te_i, e_j\rangle|^2 + \sum_{\substack{i\geqslant N+1\\ j\geqslant 1}} |\langle Te_i, e_j\rangle|^2.$$

The dominated convergence theorem shows that the right side converges to zero as $N \to \infty$, so

$$T = \sum_{i,j} \langle Te_i, e_j\rangle e_j^* \otimes e_i.$$

This shows that the $e_j^* \otimes e_i$ form an orthonormal basis for the space of Hilbert-Schmidt operators. We can obtain $e_j^* \otimes e_i = T_{f_i(x)f_j(y)}$ from unwinding the definitions. Consider the kernel

$$K(x,y) := \sum_{i,j} \langle Te_i, e_j\rangle e_i(x)e_j(y).$$

The assignment $K \mapsto T_K$ is a continuous map into the space of Hilbert-Schmidt operators (actually, the argument proves it is a bijective isometry), so we can combine the above formulas to obtain

$$T = T_K.$$