Solutions to Exercise sheet 5:

1. By Lemma 3.24 of the script if suffices to show that the second condition is equivalent to
$$\langle T_K f, f\rangle \geq 0\,.$$
Write $f = \sum a_k f_k$. Thus
$$\langle T_K f, f\rangle = \sum a_k^2 \langle T_K f_k, f_k\rangle = \sum a_k^2 \lambda_k \geq 0\,.$$
The other direction follows as
$$\lambda_k = \langle T_K f_k, f_k\rangle \geq 0\,.$$

2. Consider $\mathbb{R}^{(X)}$ and $\langle\cdot,\cdot\rangle$ as in the hint. Note that the assumptions on $K$ imply
$$\langle f, g\rangle = \langle g, f\rangle \geq 0\,.$$
We also have the Cauchy-Schwarz inequality since for any $t \in \mathbb{R}$,
$$0 \leq \langle tf + g, tf + g\rangle = t^2\langle f, f\rangle + 2t\langle f, g\rangle + \langle g, g\rangle = p(t)$$
As the quadratic polynomial $p(t)$ is always non-negative, its discriminant is non-positive: $\langle f, g\rangle^2 - \langle f, f\rangle\langle g, g\rangle \leq 0$.
However, $\mathbb{R}^{(X)}$ need not be an inner product space as one could have $\langle f, f\rangle = 0$ for a non-zero $f$ and it need not be complete.
Consider $\mathcal{C}$ to be the set of Cauchy sequences in $\mathbb{R}^{(X)}$, i.e.
$$\mathcal{C} := \{(f_n)_{n=1}^\infty : f_n \in \mathbb{R}^{(X)}, \forall\epsilon > 0 \exists N \forall m, n > N(\langle f_n - f_m, f_n - f_m\rangle < \epsilon^2)\}$$
Let $\mathcal{C}_0$ be the set of sequences in $\mathbb{R}^{(X)}$ converging to 0, i.e.
$$\mathcal{C}_0 := \{(f_n)_{n=1}^\infty : f_n \in \mathbb{R}^{(X)}, \forall\epsilon > 0 \exists N \forall n > N(\langle f_n, f_n\rangle < \epsilon^2)\}$$
For two sequences $F = (f_n)_{n=1}^\infty, G = (g_n)_{n=1}^\infty \in \mathcal{C}$ we define
$$\langle F, G\rangle = \lim_{n\to\infty} \langle f_n, g_n\rangle\,.$$
(Check that the above limit exists!)
Let us denote by $c$ the map $c : \mathbb{R}^{(X)} \to \mathcal{C}$ sending $f \in \mathbb{R}^{(X)}$ to the constant sequence $(f, f, f, ...) \in \mathcal{C}$. Note that
$$\langle f, g\rangle = \langle c(f), c(g)\rangle\,.$$
Note that $\mathcal{C}$ and $\mathcal{C}_0$ are $\mathbb{R}$-vector spaces and that $F \in \mathcal{C}_0 \iff \langle F, F\rangle = 0$. Now $(\mathcal{C}/\mathcal{C}_0, \langle\cdot,\cdot\rangle)$ forms our Hilbert space $\mathcal{H}$ as:

   (a) $\langle\cdot,\cdot\rangle$ is bilinear and symmetric by definition.

(b) $\langle \cdot, \cdot \rangle$ is well-defined as $\langle F, G \rangle = \langle F + Z_1, G + Z_2 \rangle$ for $Z_1, Z_2 \in \mathcal{C}_0$. This uses Cauchy-Schwarz.

(c) $\langle \cdot, \cdot \rangle$ is positive-definite as $\langle F, F \rangle = 0$ implies $F \in \mathcal{C}_0$ implies $F \equiv 0$ in $\mathcal{C}/\mathcal{C}_0$.

(d) $\langle \cdot, \cdot \rangle$ induces a complete norm on $\mathcal{C}/\mathcal{C}_0$.

Define $\phi : X \to \mathbb{R}^{(X)}$ by

$$x \mapsto f_x(y) := \begin{cases} 1\,, & \text{if } y = x\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

Let $\varphi$ be the composition

$$X \xrightarrow{\phi} \mathbb{R}^{(X)} \xrightarrow{c} \mathcal{C} \to \mathcal{C}/\mathcal{C}_0 = \mathcal{H}$$

Check that this works!