Exercise Sheet 7 - Solutions

1. Let $E \subset V$ be a closed subspace of a Banach space V. Prove: There exists a closed complement to E if and only if there exists a continuous linear map $P\colon V \to V$ with $P^2 = P$ and $\operatorname{im}(P) = E$.

   Solution: Let $F \subset V$ be a closed complement to E in V. The open mapping theorem proves that the map $(e, f) \in E \times F \mapsto e + v \in V$ has a continuous inverse $\varphi\colon V \to E \times F$. We construct P as the composition

   $$V \xrightarrow{\varphi} E \times F \xrightarrow{(e,f)\mapsto e} E \xrightarrow{e \mapsto e} V$$

   The composition of the first two arrows $V \to E$ is surjective, so the image of P is E. Note that P maps each $v \in V$ to the unique $e \in E$ such that there exists $f \in F$ with $f + e = v$. We get $P^2(v) - P(v) \in E \cap F$ for each $v \in V$. The subspaces E and F are complementary, so $E \cap F = 0$. This implies $P^2 = P$.

   Let $P\colon V \to V$ be a continuous, linear map with $\operatorname{im}(P) = E$. Set $F := \ker(P)$. Note that $v = P(v) + (v - P(v))$ for each $v \in V$, so $E + F = V$. Let $v \in E \cap F$ then there exists $w \in V$ with $P(w) = v$. We have the equalities

   $$v = P(w) = P^2(w) = P(v) = 0,$$

   so $E \cap F = 0$. Moreover, the subspace $F = P^{-1}(\{0\})$ is closed because P is continuous.

2. Let $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$ be Banach spaces and $T\colon V \to W$ a surjective, linear, and continuous map. Show that the following are equivalent:

   (a) The closed subspace $\ker(T)$ admits a closed complement in V.

   (b) There is a linear, continuous map $S\colon W \to V$ with $T \circ S = \operatorname{id}_W$.

   Solution: Set $E := \ker(T)$.

   Suppose F is a closed complement to E. The open mapping theorem shows that $T|_F\colon F \to W$ admits a continuous inverse $\tilde{S}\colon W \to F$. We can compose this map with the inclusion $F \to V$ to get a continuous map $S\colon W \to V$. Let $w \in W$, then $T(S(w)) = T|_F(\tilde{S}(w)) = w$.

   Let $S\colon W \to V$ be a linear, continuous map with $T \circ S = \operatorname{id}_W$. Let $F = \operatorname{im}(S)$ be a normed subspace of V. Note that S is injective, so the map $\overline{S}\colon W \to F$ is bijective. The inverse of $\overline{S}$ is given by $T|_F$, so $\overline{S}$ is a homeomorphism. This implies that F is a Banach space and hence that F is closed in V because every Cauchy sequence in F has a limit in F. We can write $v = (S(T(v)) - v) + S(T(v)) \in E + F$ for every $v \in V$. Let $v \in E \cap F$ then there is $w \in W$ with $v = S(w)$. We get

   $$v = S(w) = S(T(S(w))) = S(T(v)) = 0,$$

   so $E \cap F = 0$.

3. Show that the subspaces

$$V := \{f \in \ell^1(\mathbb{N}) : f(2n) = 0 \ \forall n \geqslant 0\}$$
$$W := \{f \in \ell^1(\mathbb{N}) : f(2n-1) = nf(2n) \ \forall n \geqslant 1\}$$

are closed in $\ell^1(\mathbb{N})$ while $V + W$ is not closed.

Hint: Show $V + W \supset c_{00}(\mathbb{N})$.

Solution: The evaluation maps $\mathrm{ev}_n(f) := f(n)$ are continuous because $|\mathrm{ev}_n(f)| \leqslant ||f||_1$ for all $n \in \mathbb{N}$. The spaces V and W are therefore intersections of kernels of continuous functionals. This implies they are closed.

Let $f \in c_{00}(\mathbb{N})$ and $N \in \mathbb{N}$ such that $f(n) = 0$ for all $n > N$. We prove by induction on N that $f \in V + W$. If $N = 0$, then f is supported at 0. In particular, $f \in W$. If N is odd then define

$$v(n) := \begin{cases} 0 & n \neq N \\ f(n) & n = N. \end{cases}$$

The function $f - v$ satisfies $(f - v)(n) = 0$ for all $n \geqslant N$, so the induction hypothesis implies $f - v \in V + W$. Because $v \in V$, this implies $f \in V + W$. If N is even and $N > 0$ then define

$$w(n) := \begin{cases} 0 & n \neq N, N-1 \\ f(n) & n = N \\ \frac{N}{2} f(n) & n = N-1. \end{cases}$$

The above proof applies almost identically to this case as well.

Suppose $V + W$ is closed then $V + W = \ell^1(\mathbb{N})$. Let $f \in \ell^1(\mathbb{N})$ then there exist unique $v \in V$ and $w \in W$ with $v + w = f$. Note that $f(2n) = w(2n)$ for all $n \in \mathbb{N}$. When we set $f(n) := \frac{1}{n^2}$ for $n \geqslant 1$ and $f(0) = 0$, then $w(2n) = \frac{1}{4n^2}$ for $n \geqslant 1$, so $w(2n-1) = \frac{1}{4n}$. This implies

$$\sum_{n \geqslant 1} \frac{1}{4n} + \frac{1}{2n^2} < \infty,$$

which is a contradiction.

4. Show that there is a bounded set function $p \colon \mathscr{P}(\mathbb{N}) \to \mathbb{R}$ such that

   (a) $p(\mathbb{N}) = 1$,

   (b) $p(A \cup B) = p(A) + p(B)$ whenever $A \cap B$ is finite.

Solution: Consider the real vector space $\ell^\infty(\mathbb{N})$ of bounded functions $\mathbb{N} \to \mathbb{R}$. Suppose we have constructed a linear map $\varphi \colon \ell^\infty(\mathbb{N}) \to \mathbb{R}$ with $|\varphi(f)| \leqslant ||f||_\infty$ for all $f \in \ell^\infty(\mathbb{N})$, and $\varphi(1_\mathbb{N}) = 1$ and $f(1_A) = 0$ for every finite subset $A \subset \mathbb{N}$. The function $p(A) := f(1_A)$ is bounded, satisfies $p(\mathbb{N}) = 1$, and for two sets $A, B \subset \mathbb{N}$ the inclusion-exclusion identity gives

$$p(A \cup B) = f(1_{A \cap B}) = f(1_A + 1_B - 1_{A \cup B}) = p(A) + p(B) - f(1_{A \cap B}).$$

In particular, when $A \cap B$ is finite this implies $p(A \cup B) = p(A) + p(B)$. Thus, it suffices to construct a functional with the above properties.

Consider the semi-norm

$$q(f) := \limsup_{n\to\infty} |f(n)|$$

and the functional $\tilde{\varphi} : \lambda 1_{\mathbb{N}} \in \mathbb{R}1_{\mathbb{N}} \mapsto \lambda \in \mathbb{R}$. The semi-norm q bounds $\tilde{\varphi}$, so by the Hahn-Banach theorem there exists an extension φ to $\ell^\infty(\mathbb{N})$ such that $|\varphi(f)| \leqslant q(f) \leqslant ||f||_\infty$ and $|\varphi(1_A)| \leqslant q(1_A) = 0$ for every finite $A \subset \mathbb{N}$.