# Exercise Sheet 8 - Solutions

1. Prove Proposition V.11 and Corollary V.12.

   *Solution:* We begin by proving Proposition V.11. Note that (i) is equivalent to (ii) because addition is continuous in topological vector spaces. Let $||\cdot||_\beta$ be a continuous seminorm on $W$, $\mathscr{F} = (||\cdot||_\alpha)_{\alpha\in\mathscr{F}}$ a set of continuous seminorms on $V$ and $r > 0$. The condition

   $$N(0;\mathscr{F};r) \subset T^{-1}(N(0;\beta;1)).$$

   is equivalent to

   $$\sup\left\{||Tx||_\beta \middle| \forall\alpha\in\mathscr{F} : ||x||_\alpha \leqslant 1\right\} < \frac{1}{r}.$$

   This shows that (ii) is equivalent to (iii).

   If we regard the topology on $\mathbb{R}$ as the topology induced by $|\cdot|$, then Proposition V.11 says that continuity of a functional $f\colon V \to \mathbb{R}$ is equivalent to the existence of a finite set of continuous seminorms $\mathscr{F}$ on $V$ with

   $$\sup\left\{||f(x)||\middle|\forall\alpha\in\mathscr{F} : ||x||_\alpha \leqslant 1\right\} < +\infty.$$

2. Let $V$ be a finite-dimensional normed space. Show that the weak topology and the norm topology on $V$ coincide.

   *Solution:* Consider a basis $(f_1,,\ldots,f_n)$ of $V^*$ and define the seminorms $||v||_i := |f(v_i)|$ for $i = 1,\ldots,n$. Any functional $f \in V^*$ is continuous with respect to the topology defined by the seminorms $||\cdot||_i$ because we can write $f = \lambda_1 f_1 + \cdots + \lambda_n f_n$ and then

   $$|f(v)| \leqslant |\lambda_1|||v||_1 + \cdots + |\lambda_n|||v||_n$$

   for all $v \in V$. This implies the weak topology is generated by the finite set $||\cdot||_i$. In particular, the weak topology is the topology defined by the norm

   $$||\cdot|| := ||\cdot||_1 + \cdots + ||\cdot||_n.$$

   Thus the weak topology coincides with the (unique) norm topology on $V$.

3. Let $X$ be a set, $\mathscr{F} = \{(\varphi, Y_i) : i \in I\}$ a family of pairs consisting of topological spaces $Y_i$ with a map $\varphi_i\colon X \to Y_i$ and equip $X$ with the initial topology with respect to $\mathscr{F}$. Prove that a sequence $(x_n)_n \in X$ converges to $x \in X$ if and only if $\varphi_i(x_n)$ converges to $\varphi_i(x)$ for all $i \in I$.

   *Solution:* Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in $X$. If the sequence has a limit point $x \in X$ then $\varphi_i(x)$ is a limit point of $(\varphi_i(x_n))_{n\in\mathbb{N}}$ for each $i \in I$ because the functions are continuous.

   For the other direction, define

   $$\mathcal{B} := \bigcup_{i\in I}\{\varphi_i^{-1}(V)|V \subset Y_i \text{ open}\} \subset \mathcal{P}(X).$$

Equip $X$ with the topology whose opens are unions of finite intersections $U_1 \cap \cdots U_n$ with $U_i \in \mathcal{B}$. This is the smallest topology which contains $\mathcal{B}$ and hence the initial topology.

Suppose there exists $x \in X$ such that $\varphi_i(x)$ is a limit point of $\varphi_i(x_n)$ for all $i \in I$. Let $W$ be a generic neighborhood of $x \in X$. There exists $i_1, \ldots, i_m \in I$ and opens $V_j \subset Y_{i_j}$ such that

$$x \in \bigcap_j \varphi_{i_j}^{-1}(V_j) \subset W$$

because such sets form a basis for the topology. There exists $N \geqslant 0$ such that $\varphi_{i_j}(x_n) \in V_{i_j}$ for each $j$ and $n \geqslant N$. In particular, we get $x_n \in W$ for all $n \geqslant N$. This implies that $x$ is a limit point of $(x_n)_n$.

4. Show that on $L^2_{\text{loc}}(\mathbb{R})$ (where we take the Lebesgue measure) there is no norm inducing the topology defined in Example V.10.

*Solution:* Suppose $L^2_{\text{loc}}(\mathbb{R})$ is a normed space with norm $||\cdot||$. Set

$$\mathscr{F} := \{||\cdot||_{L^2(K)} | K \subset \mathbb{R} \text{ compact}\}$$

then the map

$$(V, \mathscr{F}) \to (V, ||\cdot||)$$

is continuous because both sets of seminorms define the same topology. Exercise 1 implies that there is a constant $C > 0$ and compact subsets $K_1, \ldots, K_n \subset \mathbb{R}$ such that

$$||f|| \leqslant C\max\{||f||_{L^2(K_i)}|\ i = 1, \ldots, n\}.$$

Let $K = K_1 \cup \cdots \cup K_n$. The above inequalty implies

$$||\chi_{\mathbb{R}\backslash K}|| = 0$$

and therefore $\chi_{\mathbb{R}\backslash K} = 0$. But the set $K$ is compact which means $\mathbb{R}\backslash K$ is a non-empty open set and hence $\chi_{\mathbb{R}\backslash K} \neq 0$. This is the desired contradiction.