Basics on Hilbert Spaces.

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Recall that an inner product on a $\mathbb{K}$-vector space $\mathcal{H}$ is a map $\langle \, , \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ such that (1) $\forall v \in \mathcal{H}$, $u \mapsto \langle u, v \rangle$ is $\mathbb{K}$-linear.

(2) $\langle u, v \rangle = \overline{\langle v, u \rangle}$ $\forall u, v \in \mathcal{H}$.

(3) $\langle u, u \rangle \geq 0$ with equality $\iff u = 0$.

Observe that (2) automatically implies that $\langle u, u \rangle \in \mathbb{R}$ $\forall u \in \mathcal{H}$. Using then that $\forall u, v \in \mathcal{H}$ the polynomial

$$t \longrightarrow \langle u + tv, u + tv \rangle$$

is ≥ 0 one gets the

<u>Cauchy-Schwarz Inequality</u> :

$$|\langle u,v\rangle|^2 \leq \langle u,u\rangle\langle v,v\rangle \quad \forall u,v \in \mathcal{H}.$$

Defining then $\|u\| = \sqrt{\langle u,u\rangle}$ this implies :

$\|u+v\| \leq \|u\| + \|v\|$

and $\| \ \|$ is a norm on $\mathcal{H}$.

<u>Def.</u> : The inner produkt space is a Hilbert space if it is complete for $\| \ \|$.

Let $G \subset \mathcal{H}$ be any subset and $v \in \mathcal{H}$.

Define $d(v, G) := \inf \{ \|v - w\| : w \in G \}$.

<u>Thm.</u> Assume $\mathcal{H}$ is a Hilbert space and $G \subset \mathcal{H}$ is closed convex. Then $\forall v \in \mathcal{H}$ there exists a <u>unique</u> $P_G(v) \in C$

such that $\|v - P_{C'}(v)\| = d(v, C')$.

The proof is based on two facts:

(1) ▱ identity: $\forall\, v, w \in \mathcal{H}$:

$$\left\|\frac{v+w}{2}\right\|^2 + \left\|\frac{v-w}{2}\right\|^2 = \frac{1}{2}\left(\|v\|^2 + \|w\|^2\right)$$

which is a direct verification by expanding the left hand side.

(2) $\forall\, x, y, z$ in $\mathcal{H}$ let

$$c : \mathbb{R} \longrightarrow \mathcal{H}, \quad t \mapsto tx + (1-t)y$$

$$\cancel{t \longmapsto \|z - c(t)\|^2 := f(t)}$$

~~then if x~~

then if $x \neq y$, $f(t) := \|z - c(t)\|^2$

is strictly convex. Indeed:

$$f''(t) = \|x - y\|^2.$$

<u>Proof of Thm.</u>

By translating the situation by $-u$ we may assume $u = 0$. Let then $d := d(0, C)$ and $(v_n)_{n \geq 1}$ = sequence in C with $d^2 \leq \|v_n\|^2 < d^2 + \frac{1}{n}$.

By the ▭ identity we have:

$$\left\| \frac{v_n + v_m}{2} \right\|^2 + \left\| \frac{v_n - v_m}{2} \right\|^2 = \frac{1}{2}\left(\|v_n\|^2 + \|v_m\|^2\right).$$

~~But:~~

$$< d^2 + \frac{1}{2}\left(\frac{1}{n} + \frac{1}{m}\right).$$

Now $\left\| \frac{v_n + v_m}{2} \right\|^2 \geq d^2$ since $\frac{v_n + v_m}{2} \in C$

and hence:

$$\left\| \frac{v_n - v_m}{2} \right\|^2 < \frac{1}{2}\left(\frac{1}{n} + \frac{1}{m}\right)$$

which implies that $(v_n)_{n \geq 1}$ is Cauchy.

Since $\mathcal{H}$ is Hilbert $x := \lim\limits_{n \to \infty} v_n \in C$

exists and clearly

$$d(0, C) = \|x\|.$$

Uniqueness follows immediately from the strict convexity property (2) in H3. $\boxed{\S}$

Let now $A \subset \mathcal{H}$ be a subset of an inner product space. We define:

$$A^{\perp} := \{ v \in \mathcal{H} : \langle a, v \rangle = 0 \quad \forall a \in A \}.$$

Clearly $A^{\perp}$ is a closed $\mathbb{K}$-vector subspace of $\mathcal{H}$.

Observe the following (immediate) properties:

(1) $(A^{\perp})^{\perp} \supset A$.

(2) $(\overline{A})^{\perp} = A^{\perp}$, where $\overline{A}$ is the closure of A.

We have then:

<u>Prop.</u> Let $V \subset \mathcal{H}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ is the orthogonal direct sum $\mathcal{H} = V \oplus V^{\perp}$

of V and $V^\perp$.

This means that every $v \in \mathcal{H}$ is uniquely the sum $\quad v = v_1 + v_1^\perp, \; v_1 \in V, v_1^\perp \in V^\perp$

and $\|v\|^2 = \|v_1\|^2 + \|v_1^\perp\|^2$.

<u>Proof</u>: Since V is closed and convex we have the nearest point projection

$$P_V : \mathcal{H} \to V$$

from Thm H2. ~~Then, $\forall$ u~~ Let $v \in V$; then $\forall u \in V$ the polynomial

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \|v - (P_V(u) + tu)\|^2 \end{aligned}$$

attains a minimum at $t = 0$. Hence the coefficient of the linear factor must vanish:

$$\operatorname{Re}\langle v - P_V(v), u\rangle = 0 \quad \forall u \in V.$$

Now let $u \in V$ and pick $\alpha \in \mathbb{C}$, $|\alpha| = 1$ such that

$$\left| \langle v - P_V(v), u \rangle \right| = \alpha \cdot \langle v - P_V(v), u \rangle$$

$$= \operatorname{Re} \langle v - P_V(v), \bar{\alpha} u \rangle$$

$$= 0$$

since $\bar{\alpha} u \in V$. Thus $v - P_V(v) \in V^{\perp}$

and $\|v\|^2 = \| P_V(v) + (v - P_V(v)) \|^2$

$$= \| P_V(v) \|^2 + \| v - P_V(v) \|^2 \qquad \boxed{\Xi}$$

From this we deduce:

<u>Thm.</u> (Riesz Repr.) The map $i : \mathcal{H} \to \mathcal{H}^*$ which to $u \in \mathcal{H}$ associates $i(u) \in \mathcal{H}^*$ defined by $i(u)(v) = \langle v, u \rangle$ is an antilinear, norm preserving, bijection.

<u>Proof</u>: Antilinearity is clear; for the norm we have $\|i(u)\| = \sup_{\|v\| \leq 1} |\langle v, u\rangle| \leq \|u\|$ by C.S. Since $i(u)\left(\frac{u}{\|u\|}\right) = \frac{\|u\|^2}{\|u\|} = \|u\|$ the map i is norm preserving.

Let now $\lambda \in \mathcal{H}^*$; we may assume $\lambda \neq 0$. Then $\operatorname{Ker}\lambda$ is a closed, codimension 1 subspace of $\mathcal{H}$. By Prop. H-5

$$\operatorname{Ker}\lambda \oplus (\operatorname{Ker}\lambda)^\perp = \mathcal{H}.$$

Let $(\operatorname{Ker}\lambda)^\perp = \mathbb{K}\cdot e$ with $\|e\| = 1$.

Let $u := \overline{\lambda(e)} \cdot e$. Then: $v = v_1 + r\cdot e$ $r \in \mathbb{K}$, $v_1 \in \operatorname{Ker}\lambda$ and:

$$\langle v, \underbrace{\overline{\lambda(e)}\cdot e}_{u} \rangle = \langle re, \overline{\lambda(e)}\cdot e\rangle = r\lambda(e)$$

$$= \lambda(re) = \lambda(v)$$

which shows $i(u) = \lambda$. $\boxed{\S}$