1. Banach spaces, bounded linear maps.

$(V, \| \ \|)$: normed space, V a $\mathbb{K}$-v.s., $\mathbb{K} = \mathbb{R}, \mathbb{C}$. $\Rightarrow$ V metric $\Rightarrow$ V topology.

Then: $V \times V \xrightarrow{+} V$, $\mathbb{K} \times V \xrightarrow{\cdot} V$ are cont.

$\Rightarrow$ Concept of TVS.

Banach = complete normed space.

Thm 1.18 : $T : V \to W$ linear :

T cont. $\Leftrightarrow$ T bounded $\Leftrightarrow$ $\|T\| := \sup\limits_{\|x\| \leq 1} \|T(x)\| < +\infty$.

$\Rightarrow$ Norm on $\mathcal{B}(V, W)$; dual $V^* = \mathcal{B}(V, \mathbb{K})$.

Examples :

(1) Hilbert spaces, see "Basics of Hilbert Spaces".

(2) $L^p(\Omega, \mu)$ where $(\Omega, \mathcal{F}, \mu)$ measure space, $1 \leq p \leq +\infty$. Then : isometric bij.

$$L^q(\Omega, \mu) \to L^p(\Omega, \mu)^*, \quad \frac{1}{p} + \frac{1}{q} = 1$$

$1 \leq p < +\infty$.

(3) X top. space, $C_b(X, \mathbb{K}) = \{ f : X \to \mathbb{K}$ cont. bnded $\}$

$$\|f\|_b = \sup_{x \in X} |f(x)|.$$

X loc. c. Hausdorff,

$C_0(X) = \{ f : X \to \mathbb{K} : f$ cont. vanish at $\infty \}$.

Closed subspace of $C_b(X, \mathbb{K})$.

$C_0(X, \mathbb{C})^* \simeq M(X, \mathbb{C})$ space of complex measures [Ex. 2.16]

$C_0(X, \mathbb{R})^* \simeq M(X, \mathbb{R})$ space of signed measures. [Ex. 2.16]

+ Rudin Chapt. 6.

(4) W Banach $\Rightarrow$ $\mathcal{B}(V, W)$ is Banach.

(5) $c_0(\mathbb{N})$, $\ell^1(\mathbb{N})$, $\ell^\infty(\mathbb{N})$

$c_0(\mathbb{N})^* = \ell^1(\mathbb{N})$, $\ell^1(\mathbb{N})^* = \ell^\infty(\mathbb{N})$.

(6) $\Omega \overset{\text{open}}{\subset} \mathbb{R}^n$, $W^{p,k}(\Omega) = \{ f \in L^1_{loc}(\Omega) :$

$D^\alpha_w f \in L^p(\Omega), \forall |\alpha| \le k \}$.

$$\|f\|_{p,k} = \sum_{|\alpha| \le k} \|D^\alpha_w f\|_p .$$

(7) $C^r_b(\Omega) = \{ f : \Omega \to \mathbb{C} : D^\alpha f$ is cont. and bounded $\forall |\alpha| \le k \}$

$$\|f\| = \sum_{|\alpha| \le k} \|D^\alpha f\|_b .$$

The relation between $W^{2,k}(\mathbb{R}^n)$ and $C^r_b(\mathbb{R}^n)$ is the topic of Chapter 7.

If $T \in \mathcal{B}(V, W)$ then $T^* \in \mathcal{B}(W^*, V^*)$ and

H.B. (chap. 2) $\Rightarrow \|T^*\| = \|T\|$.

Special Classes:

(1) $T : V \to W$, isometry.

(2) $T : \mathcal{H} \to \mathcal{H}$, Hilbert — Self-Adjoint, Unitary.

(3) Mult. Oper. $(\Omega, \mathcal{F}, \mu)$ measure space

$$M_\varphi : L^p(\Omega) \to L^p(\Omega)$$
$$f \mapsto \varphi \cdot f$$

$\| M_\varphi \| = \| \varphi \|_\infty$.

(4) Γ group : $\Gamma \xrightarrow{\lambda} U(\ell^2(\Gamma))$, Unit. Rep.

(5) Integral op. $(\Omega, \mathcal{F}, \mu)$ σ-finite.

$K \in L^2(\Omega \times \Omega, \mathbb{K})$: $T_K f(x) = \int K(x,y) f(y) \, d\mu(y)$

$$T_K : L^2(\Omega) \to L^2(\Omega),$$

$$\| T_K \| \leq \| K \|_2$$

2. Hahn-Banach.

V $\mathbb{R}$-v.s.

Def. 2.1 (Gauge) $p : V \to \mathbb{R}$.

Thm 2.4. (HB)

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook] & \mathbb{R} \\ V \arrow[ur, dashed, "\exists F"'] & \end{tikzcd}$$

lin, $f \leq p$.

$F \leq p$.

Version $V = \mathbb{C}$-v.s. and gauge is replaced by a semi-norm.

Various consequences:

- $(V, \| \ \|)$ normed v.s. $f \in M^*$ then $\exists F \in V^*$, $\|F\| = \|f\|$ (Cor. 2.8)

- $\|v\| = \sup \{ |f(v)| : f \in V^*, \|f\| \leq 1 \}$.

3. Compact Operators. Spectral thm.

Def. 3.1. $T: V \to W$ is compact if $\overline{T(B_{\leq 1}(0))}$ is compact.

Example: if $\|T - T_n\| \to 0$ and T_n finite rank.

And:

Def. 3.6. $T: \mathcal{H} \to \mathcal{H}$ HS if

$$\|T\|_2 := \sum_{n,m=1}^{\infty} |\langle Te_n, e_m\rangle|^2 < +\infty .$$

Then $\|T\| \leq \|T\|_2$.

H.S $\Rightarrow$ Compact.

Example: $K \in L^2(\Omega \times \Omega, \mu \times \mu)$ then T_K is HS and $\|T_K\|_2 = \|K\|_2$.

Spectral Thm (3.14) $T: \mathcal{H} \to \mathcal{H}$ compact self-adjoint:

(1) $\mathcal{H}$ has a ONB of eigenv. and all e.v. are real.

(2) $\forall \lambda \neq 0$ $\dim \mathcal{H}_\lambda < +\infty$ and $\forall \varepsilon > 0$

$\{ \lambda \in \mathbb{R} : |\lambda| \geq \varepsilon, \ \dim \mathcal{H}_\lambda > 0 \}$ is finite.

In particular, applies to T_K, $K(x,y) = \overline{K(y,x)}$.

Mercer's thm: if K is continuous, semi-positive definite.

Example 3.17: Unitary reps. of compact grps.

4. Baire Category and Consequences.

Thm. (see Thm 4.5) (X,d) complete metric

$$X = \bigcup_{n \in \mathbb{N}} F_n, \ F_n \text{ closed} \Rightarrow \exists n_0, \ \overset{\circ}{F}_{n_0} \neq \emptyset.$$

Thm. 4.14. (Uniform boundedness) V Banach, W normed.

$T_\lambda \in \mathcal{B}(V,W)$, $\lambda \in \Lambda$, with $\sup_{\lambda \in \Lambda} \|T_\lambda(v)\|_W < +\infty$ $\forall v \in V$

Then $\sup_{\lambda \in \Lambda} \|T_\lambda\| < +\infty$.

Thm 4.19 (Open mapping) V and W Banach

$T : V \to W$ bounded surj. Then T is open.

Cor: T bij. $\Rightarrow$ $T^{-1} : W \to V$ bnded.

Thm 4.22 (Closed Graph) V, W Banach, $T : V \to W$ linear. Assume $\Gamma = \{(v, T(v)) : v \in V\} \subset V \times W$ closed. Then T is bnded.

- Grothendieck's thm. on closed subsp. of L^r.
- Non existence of closed compl. $c_0(\mathbb{N})$ in $\ell^\infty(\mathbb{N})$.

5. TVS, weak topologies, Banach–Alaoglu.

Concept: Topology generated by a family of semi-norms $\{\| \ \|_\alpha : \alpha \in \mathcal{A}\}$,

$\| \ \|_\alpha : V \to \mathbb{R}_{\geq 0}$ seminorm.

Sufficient if $\forall v \neq 0 \ \exists \alpha \in A, \|v\|_\alpha \neq 0$.

$\Leftrightarrow$ V Hausdorff.

<u>Examples</u> 5.9. X l.c. H : on $C(X)$, suff. family of seminorms $\| \ \|_K$, $K \subset X$ compact.

<u>Example 5.10.</u> $(X, \mathcal{F}, \mu)$, $L^p_{loc}(X)$,

$\|f\|_{1,K}$, $K \subset X$ compact.

- Characterization of continuous linear maps.

<u>Thm 5.13</u>. (H-B) $\{\| \ \|_\alpha : \alpha \in A\}$ sufficient family of seminorms : $\forall v \neq 0 \ \exists F \in V^*$ $F(v) \neq 0$.

Let V be a TVS (main example: V normed).

- $\sigma(V, V^*)$ – top. on V
- $\sigma(V^*, V)$ – top. on V^*

$\Big\}$ in terms of seminorms.

- $\sigma(V, V^*)$ top. on V: initial topology for

$\mathcal{F} = \{ (\lambda, \mathbb{K})_{\lambda \in V^*} \}$

- $\sigma(V^*, V)$ top. on V^*: initial topology for

$\{ (J(v), \mathbb{K}) : v \in V \}$ where $J(v)(f) = f(v)$.

Thm 5.29 (Banach–Alaoglu) $(V, \| \cdot \|)$ normed.
Then $B^{V^*}_{\leq 1}(0)$ is weak $*$-compact.

Cor. 5.31 X compact Hausdorff. Then the space $M^1(X)$ of probability measures is weak*-compact, convex.

And $\forall \varphi \in \mathcal{H}omeo(X)$ the map

$$\begin{aligned} M^1(X) &\to M^1(X) \quad \text{is weak}^*\text{-cont.} \\ \mu &\mapsto \varphi_*(\mu) \end{aligned}$$

6. Convexity, Markoff-Kakutani, Krein-Milman.

V $\mathbb{R}$-vector space.

Concept: Convex subset.

The following are also called "Geometric forms of H.B.":

<u>Thm 6.8</u> V, $\{\| \|_\alpha : \alpha \in \mathcal{A}\}$. $A \subset V$ $\neq \emptyset$, open, convex, $v \notin A$: $\exists F \in V^*$

$$F(a) < F(v) \quad \forall a \in A.$$

<u>Cor. 6.9</u> $A \subset V$, $\neq \emptyset$, closed, convex, $v \notin A$. $\exists \alpha \in \mathbb{R}$, $\exists F \in V^*$:

$$F(a) < \alpha < F(v) \quad \forall a \in A.$$

<u>Kakutani-Markov F.P.T</u>: $G \longrightarrow Aut(V)$ G abelian, $A \subset V$ ~~conv~~ $\neq \emptyset$, compact, convex, G-inv. $\Rightarrow \exists$ G-fixed point in A.

Applications: – existence of invariant measures
[Cor. 6.21 + Cor. 6.22]

– amenable groups. [see additional notes].

See also Thm 6.25 on uniform distribution.

Thm 6.28 (Krein–Milman) V, $\{\|\ \|_\alpha : \alpha \in A\}$ sufficient. $A \subset V$ convex, compact,

$$A = \overline{\mathrm{conv}(ex(A))}.$$

Application: Existence of ergodic invariant measures.

[Def 6.30, Cor. 6.32].

7. Fourier Analysis & Sobolev.

The main thm. of this chapter is:

Thm 7.30. Assume $k, r \in \mathbb{N}$ and $k > r + \frac{n}{2}$

Then if $f \in W^{2,k}(\mathbb{R}^n)$, $f \in C_b^r(\mathbb{R}^n)$

and the inclusion $W^{2,k}(\mathbb{R}^n) \longrightarrow C_b^r(\mathbb{R}^n)$

is bounded.

For the definition of $W^{p,k}(\Omega)$ one needs

the concept of weak derivative (Def. 7-21)

$D_W^\alpha f$ for $f \in L^1_{loc}(\Omega)$ and its uniqueness,

when it exists (lemma 7-22).

The two major tools are

- Fourier Analysis on $\mathbb{R}^n$
- Convolution.

Concerning Fourier Analysis, which is part of Analysis IV, one establishes

<u>Thm 7.10.</u> Fourier inversion formula for $f \in C_{00}^{\infty}(\mathbb{R}^n)$.

<u>Thm 7.11.</u> Plancherel: $\mathcal{F} : L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n)$

takes values in $L^2(\mathbb{R}^n)$ and extends to a <u>unitary</u> operator $\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$.

Important tools are interaction between the D^α-operators, Fourier transform, convolution.
In particular approximation of identity play a crucial role.