7. Fourier Analysis and the Sobolev Embedding Thms

Concentrate on $W^{2,k}(\mathbb{R}^n)$ = functions on $\mathbb{R}^n$ which are L^2 and admit "weak derivatives" in L^2 up to order k. Major point is to show that if $k > r + \frac{n}{2}$ this space consists of bnded C^r functions.

7.1. Basic Fourier Analysis on $\mathbb{R}^n$.

Let $\mathcal{L}$ = Lebesgue measure on $\mathbb{R}^n$, normalised so that $\mathcal{L}([0,1]^n) = 1$ and $m = (2\pi)^{-n/2}\mathcal{L}$. Then in the sequel $L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n, m)$;

$$\langle x, \xi \rangle := \sum_{i=1}^{n} x_i \xi_i \qquad x, \xi \in \mathbb{R}^n$$

$$\langle f, g \rangle := \int f \bar{g} \, dm \quad \text{if} \quad |f \cdot g| \in L^1(\mathbb{R}^n).$$

All our function spaces are $\mathbb{C}$-valued.

<u>Def. 7.1.</u> For $f \in L^1(\mathbb{R}^n)$ its Fourier transform, denoted $\hat{f}$ or $\mathcal{F} f$ is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\langle x, \xi \rangle} \, dm(x).$$

Observe that this makes perfect sense since

$$|f(x) e^{-i\langle x, \xi \rangle}| = |f(x)| \quad \forall x \in \mathbb{R}^n$$

and $f \in L^1(\mathbb{R}^n)$.

Recall $C_0(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{C} :$ f cont., vanishes at $\infty \}$, with $\| \ \|_b$ sup norm.

<u>Prop. 7. 2</u> If $f \in L^1(\mathbb{R}^n)$ then $\hat{f} \in C_0(\mathbb{R}^n)$ and $\mathcal{F} : L^1(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n)$ has
$\|\mathcal{F}\| \leq 1$.

Let $(\lambda(x) f)(y) = f(y - x)$; the translation operator. Then

<u>Lemma 7. 3.</u> $1 \leq p < +\infty$, $f \in L^p(\mathbb{R}^n)$

Then:
$$\begin{aligned} \mathbb{R} &\longrightarrow L^p(\mathbb{R}^n) \\ x &\longmapsto \lambda(x) f \end{aligned}$$
is continuous.

<u>Rem. 7. 4,</u> for $p = +\infty$, $x \mapsto \lambda(x) f$ is cont. $\Leftrightarrow$ $f \equiv$ uniformly cont. fct on $\mathbb{R}^n$!

Proof of Lemma 7.3:

From $\| \lambda(x) f - \lambda(y) f \|_p = \| \lambda(x-y) f - f \|_p$

suffices to show cont. at $x = 0$.

First: $\varphi \in C_{00}(\mathbb{R}^n)$, $\operatorname{supp} \varphi \subset B_{\leq R}(0)$,

$x \in B_{\leq 1}(0)$. Then:

$$\| \lambda(x) \varphi - \varphi \|_p^p = \int_{B_{\leq R+1}(0)} | \varphi(y-x) - \varphi(y) |^p \, dm(y)$$

$$\leq \Big(\sup_{y \in B_{\leq R+1}(0)} | \varphi(y-x) - \varphi(y) |^p \Big) \, m(B_{\leq R+1}(0))$$

$\longrightarrow 0$ with $x \to 0$ by uniform cont.

Now let $f \in L^p(\mathbb{R}^n)$ and $\varepsilon > 0$: since

$p < +\infty$ $\exists \varphi \in C_{00}(\mathbb{R}^n)$

$$\| f - \varphi \|_p < \varepsilon .$$

Let $\delta > 0$ $\| \lambda(x) \varphi - \varphi \|_p < \varepsilon$ $\forall x \in B_{\leq \delta}(0)$.

Then:

$$\|\lambda(x)f - f\|_p \leq \|\lambda(x)f - \lambda(x)\varphi\|_p + \|\lambda(x)\varphi - \varphi\|_p + \|\varphi - f\|_p$$

$$\leq 3\varepsilon. \qquad \boxed{\xi}$$

Proof of Prop. 7.2.

(1) Continuity:

$$\hat{f}(\xi+h) - \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-i\langle x,\xi\rangle}\left[e^{-i\langle x,h\rangle} - 1\right] dm(x)$$

Let $\varepsilon > 0$ and $R > 0$ with

$$\int_{\mathbb{R}^n \setminus B_{\leq R}(0)} |f| < \varepsilon.$$

Then:

$$|\hat{f}(\xi+h) - \hat{f}(\xi)| \leq \int_{B_{\leq R}(0)} |f(x)|\, \left|e^{-i\langle x,h\rangle} - 1\right| dm(x)$$

$$+ 2\varepsilon$$

$$\leq \|f\|_1 \sup_{x \in B_{\leq R}(0)} |e^{-i\langle x, h\rangle} - 1| + 2\varepsilon.$$

But for R fixed $e^{-i\langle x, h\rangle} - 1 \xrightarrow{h \to 0} 0$ unif. on $B_{\leq R}(0)$.

(2) $|\hat{f}(\xi)| \leq \|f\|_1 \Rightarrow \|\mathcal{F}\| \leq 1.$

(3) (Riemann - Lebesgue)

In $\hat{f}(\xi) = \int f(x)\, e^{-i\langle x, \xi\rangle}\, dm(x)$

$$x \longmapsto x - \frac{\pi \xi}{\|\xi\|^2}$$

to get:

$$= \int f\left(x - \frac{\pi \xi}{\|\xi\|^2}\right) e^{-i\left\langle x - \frac{\pi\xi}{\|\xi\|^2}, \xi\right\rangle} dm$$

$$= \int \left(\lambda\left(\frac{\pi \xi}{\|\xi\|^2}\right) f\right)(x)\, e^{-i\langle x, \xi\rangle} \underbrace{e^{i\pi}}_{-1}\, dm(x)$$

$$\hat{f}(\xi) = \int f(x)\, e^{-i\langle x,\xi\rangle}\, dm(x)$$

$$= -\int \left(\lambda\left(\tfrac{\pi\xi}{\|\xi\|^2}\right) f\right)(x)\, e^{-i\langle x,\xi\rangle}$$

$$\Rightarrow \quad 2\hat{f}(\xi) = \int \left(f - \lambda\left(\tfrac{\pi\xi}{\|\xi\|^2}\right) f\right)(x)\, e^{-i\langle x,\xi\rangle}\, dm(x)$$

$$2\,|\hat{f}(\xi)| \leq \left\| f - \lambda\left(\tfrac{\pi\xi}{\|\xi\|^2}\right) f \right\|_1$$

But as $\xi \to +\infty$, $\frac{\pi\xi}{\|\xi\|^2} \longrightarrow 0$. $\boxed{\xi}$

One major problem with the Fourier transform is that for general $f \in L^1(\mathbb{R}^n)$, $\hat{f}$ does not have any global integrability properties. Indeed one can find $f \in C_{00}(\mathbb{R}^n)$, that is continuous with compact support such that $\hat{f} \notin L^1(\mathbb{R}^n)$; it is an instructive exercise to find an example.

The next proposition shows that if one adds local regularity properties on f then $\hat{f} \in L^p$. To this end, recall that given a multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ D^α denotes the differential operator,

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and for $k \geq 1$,

$$C^k(\mathbb{R}^n) = \{ f : \mathbb{R}^n \to \mathbb{C} : D^\alpha f$$

exists and is continuous $\forall \alpha, |\alpha| \leq k \}$.

Also for $\xi \in \mathbb{R}^n$, let $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

~~Then we have:~~

Also def. : $C^k_{oo}(\mathbb{R}^n) = C^k(\mathbb{R}^n) \cap C_{oo}(\mathbb{R}^n)$.

Prop. 7.5.

(1) $f \in C^1_{00}(\mathbb{R}^n)$, then:

$$\widehat{\left(\frac{\partial f}{\partial x_j}\right)}(\xi) = i \cdot \xi_j \hat{f}(\xi) \qquad 1 \leq j \leq n.$$

(2) $f \in C^k_{00}(\mathbb{R}^n)$, then

$$\widehat{D^\alpha f}(\xi) = i^{|\alpha|} \xi^\alpha \hat{f}(\xi), \ \forall \ |\alpha| \leq k.$$

(3) $f \in C^\infty_{00}(\mathbb{R}^n)$, then

$$\hat{f} \in C_0(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \qquad \forall p \geq 1.$$

Proof:

(1) $$\widehat{\left(\frac{\partial f}{\partial x_j}\right)}(\xi) = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x)\, e^{-i\langle x, \xi \rangle}\, dm(x)$$

$$= - \int_{\mathbb{R}^n} f(x) \underbrace{\frac{\partial}{\partial x_j}\left(e^{-i\langle x, \xi \rangle}\right)}_{-i \xi_j e^{-i\langle x, \xi \rangle}} dm(x)$$

$$= i\xi_j \hat{f}(\xi).$$

(2) Follow from (1) by induction.

(3) From (2) we get that $\xi^\alpha \hat{f}(\xi)$ is bounded $\forall \alpha \in \mathbb{N}^n$ and hence so is

$$\prod_j (1+\xi_j^2)\, \hat{f}(\xi).$$

But then: $$|\hat{f}(\xi)| \leq \frac{C}{\prod_j (1+\xi_j^2)}$$

and the right hand side is obviously L^p $\forall p \geq 1$. $\boxed{\xi}$

Observe that (2) says that the Fourier transf. "converts" the diff. operator D^α into the multiplication op. ξ^α.

Next, for $a \neq 0$ in $\mathbb{R}$ let

$$(S_a f)(x) = f\left(\frac{x}{a}\right)$$

and recall that we had defined $\forall x \in \mathbb{R}^n$

$$(\lambda(x) f)(y) = f(y - x).$$

Then let's examine the behaviour under Fourier transform:

__Lemma 7-6.__ Let $f \in L^1(\mathbb{R}^n)$.

(1) $\widehat{\lambda(x) f}(\xi) = e^{-i\langle x, \xi\rangle} \hat{f}(\xi)$

(2) $(\lambda(x)\hat{f})(\xi) = \left(\widehat{e^{i\langle x, \cdot\rangle} \cdot f}\right)(\xi)$

(3) $(\widehat{S_a f})(\xi) = a^n (S_{1/a} \hat{f})(\xi).$

Proof :

(1) $$\widehat{(\lambda(x) f)}(\xi) = \int_{\mathbb{R}^n} f(y-x)\, e^{-i\langle y, \xi\rangle}\, dm(y)$$

$$= \int_{\mathbb{R}^n} f(y)\, e^{-i\langle y+x, \xi\rangle}\, dm(y)$$

$$= e^{-i\langle x, \xi\rangle}\, \hat{f}(\xi).$$

(2) Similar computation.

(3) $$\widehat{(S_a f)}(\xi) = \int_{\mathbb{R}^n} f\left(\frac{x}{a}\right) e^{-i\langle x, \xi\rangle}\, dm(x)$$

$$= \int_{\mathbb{R}^n} f(y)\, e^{-i\langle ay, \xi\rangle}\, a^n\, dm(y)$$

$$= a^n \left(S_{1/a} \hat{f}\right)(\xi). \qquad \boxed{\xi}$$

The following computation of Fourier transform is a classical application of complex analysis [see Iacobelli Example 3.12]

Example 7.7 Let $\varphi(x) = e^{-\frac{\|x\|^2}{2}}$

then $\hat{\varphi} = \varphi$.

This fact will be used in the Fourier inversion formula.

We now define, what will turn out to be the inverse Fourier transform:

Def. 7-8 For $h \in L^1(\mathbb{R}^n)$ define

$$\tilde{h}(x) = \int_{\mathbb{R}^n} f(\xi)\, e^{i\langle x, \xi \rangle}\, dm(\xi).$$

We will also use the notation

$\mathcal{F}^* h = \tilde{h}$, which is justified by

Lemma 7.9 If $f \in L^1(\mathbb{R}^n)$ and $h \in L^1(\mathbb{R}^n)$ then $\langle \mathcal{F}(f), h \rangle = \langle f, \mathcal{F}^*(h) \rangle$

Proof:

$$\langle \mathcal{F}(f), h \rangle = \int_{\mathbb{R}^n} \hat{f}(\xi) \overline{h(\xi)} \, dm(\xi)$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) e^{-i \langle x, \xi \rangle} dm(x) \overline{h(\xi)} \, dm(\xi)$$

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}^n} dm(x) f(x) \underbrace{\int_{\mathbb{R}^n} dm(\xi) e^{-i \langle x, \xi \rangle} \overline{h(\xi)}}_{\overline{\int_{\mathbb{R}^n} dm(\xi) e^{i \langle x, \xi \rangle} h(\xi)}}$$

$$= \langle f, \mathcal{F}^*(h) \rangle . \quad \boxed{\xi}$$

Now we proceed to show a version of the Fourier inversion formula. [See Thm 3.25 in Iacobelli.]

<u>Thm 7.10.</u> For $f \in C_{00}^{\infty}(\mathbb{R}^n)$, we have

$$\mathcal{F}^{*}\mathcal{F} f = f.$$

<u>Proof:</u> Observe that since $f \in C_{00}^{\infty}(\mathbb{R}^n)$ we have $\mathcal{F} f \in L^1(\mathbb{R}^n)$ by Prop 7.5(3), so $\mathcal{F}^{*}\mathcal{F} f$ is well defined. First we claim that it suffices to show that

$$\mathcal{F}^{*}\mathcal{F} f(0) = f(0) \quad \forall f \in C_{00}^{\infty}(\mathbb{R}^n).$$

Indeed, assuming this we have $\forall x \in \mathbb{R}^n$

$$f(x) = (\lambda(-x) f)(0) = (\mathcal{F}^{*}\mathcal{F}\, \lambda(-x) f)(0)$$

$$= \mathcal{F}^{*}\left(e^{i\langle x, \cdot\rangle} \mathcal{F} f\right)(0) \qquad (\text{lemma } 7.6\,(1))$$

$$= (\mathcal{F}^{*}\mathcal{F} f)(x) \quad \text{by definition.}$$

Now we turn to the proof of $\check{\mathcal{F}}\hat{\mathcal{F}}f(0) = f(0)$

which is

$$f(0) = \int_{\mathbb{R}^n} \hat{f}(\xi)\, dm(\xi).$$

This can be written

$$f(0) = \langle \hat{f}, \mathbb{1} \rangle.$$

Now let $\varphi \in C^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ s.t.

$$\begin{cases} 0 \leq \varphi \leq 1 & \varphi(0) = 1 \\ \hat{\varphi} \in L^1(\mathbb{R}^n) \end{cases}$$

We will apply this later to $\varphi(x) = e^{-\|x\|^2/2}$.

Recall $(S_a \varphi)(\xi) = \varphi\left(\frac{\xi}{a}\right)$ $a \neq 0$.

Now $S_a \varphi \xrightarrow{a \to +\infty} \mathbb{1}$ uniformly on compact sets of $\mathbb{R}^n$, together with

$$0 \leq S_a \varphi \leq 1$$

implies $\langle \hat{f}, S_a \varphi \rangle \xrightarrow[a \to +\infty]{} \langle \hat{f}, \mathbb{1} \rangle$.

Now we compute $\langle \hat{f}, S_a \varphi \rangle$ in a different way:

$$\langle \hat{f}, S_a \varphi \rangle = \langle \mathcal{F} f, S_a \varphi \rangle \overset{\text{Lemma 7.9}}{=} \langle f, \overset{*}{\mathcal{F}} S_a \varphi \rangle$$

$$= \langle f, a^n S_{1/a} \overset{*}{\mathcal{F}} \varphi \rangle$$

$$= \int_{\mathbb{R}^n} f(x)\, a^n\, \overline{\overset{*}{\mathcal{F}} \varphi (a \cdot x)}\, dm(x)$$

$$= \int_{\mathbb{R}^n} f\left(\frac{x}{a}\right) \overline{\overset{*}{\mathcal{F}} \varphi (x)}\, dm(x)$$

$$= \langle S_a f, \overset{*}{\mathcal{F}} \varphi \rangle.$$

Now: $S_a f \xrightarrow{a \to +\infty} f(0)$ uniformly on compact subsets of $\mathbb{R}^n$. In addition

$$\| S_a f \|_b = \| f \|_b$$

Since $(\mathcal{F}^* \varphi)(x) = \hat{\varphi}(-x)$, $\mathcal{F}^* \varphi \in L^1(\mathbb{R}^n)$

and hence

$$\lim \langle S_a f, \mathcal{F}^* \varphi \rangle = f(0) \int_{\mathbb{R}^n} \overline{\mathcal{F}^* \varphi(x)}\, dm(x).$$

This shows:

$$\langle \hat{f}, 1\!\!1 \rangle = f(0) \int_{\mathbb{R}^n} \overline{\mathcal{F}^* \varphi(x)}\, dm(x)$$

Finally

$$\overline{\mathcal{F}^* \varphi(x)} = \overline{\int_{\mathbb{R}^n} \varphi(\xi) e^{i\langle x, \xi \rangle} dm(\xi)}$$

$$= \int_{\mathbb{R}^n} \overline{\varphi(\xi)}\, e^{-i\langle x, \xi \rangle} dm(\xi)$$

For $\varphi(\xi) = e^{-\frac{\|\xi\|^2}{2}}$ this equals

$$= \hat{\varphi}(x) = \varphi(x)$$

hence $\displaystyle\int_{\mathbb{R}^n} \overline{\mathcal{F}^* \varphi(x)}\, dm(x) = \int_{\mathbb{R}^n} e^{-\frac{\|x\|^2}{2}} dm(x)$

$$= \hat{\varphi}(0) = \varphi(0) = 1.$$

$\boxtimes$

From Thm 7.10 and lemma 7.9 we conclude:

<u>Cor. 7.11</u> $\langle \mathcal{F}f, \mathcal{F}g \rangle = \langle f, g \rangle \quad \forall f, g \in C_{00}^{\infty}(\mathbb{R}^n)$.

<u>Proof</u>: $\langle \mathcal{F}f, \mathcal{F}g \rangle \underset{7.9}{=} \langle f, \mathcal{F}^{*}\mathcal{F}g \rangle \underset{7.10.}{=} \langle f, g \rangle$

$\boxed{?}$

In particular $\|\mathcal{F}f\|_2 = \|f\|_2 \quad \forall f \in C_{00}^{\infty}(\mathbb{R}^n)$.

In fact:

<u>Thm 7.12</u> (Plancherel) The map

$$\mathcal{F} : L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n)$$

takes values in $L^2(\mathbb{R}^n)$ and extends uniquely to a unitary operator

$$\mathcal{F} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n).$$

For the proof we will need the following two facts:

Lemma 7.13. Let V, W be Banach spaces, $D \subset V$ a vector subspace and $T : D \to W$ a bounded linear operator. Then T extends uniquely to a bounded linear operator

$$T_{ext} : \overline{D} \to W.$$

If in addition, $\|T(v)\|_W = \|v\|_V \quad \forall v \in D$ then $\|T_{ext}(v)\|_W = \|v\|_V \quad \forall v \in \overline{D}$.

Proof: Left as an exercise. $\square$

The other fact is the content of lemma 7.20 to be proven in the next section:

"Lemma 7.20". Let $1 \leq p_1, p_2 < +\infty$ and $f \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$. Then there is a sequence $(\varphi_k)_{k \geq 1}$ in $C_{00}^{\infty}(\mathbb{R}^n)$ such that

$$\| \varphi_k - f \|_{p_1} \longrightarrow 0$$

$$\| \varphi_k - f \|_{p_2} \longrightarrow 0.$$

<u>Proof of Plancherel:</u>

Apply lemma 7.13 to $V = W = L^2(\mathbb{R}^n)$ and $D = C_{00}^{\infty}(\mathbb{R}^n)$ to conclude that

$$\mathcal{F} : C_{00}^{\infty}(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

extends uniquely to an isometry

$$\mathcal{F}_{ext} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n).$$

As for the density of $C_{00}^{\infty}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ one can either use Stone-Weierstrass skillfully or lemma 7.20 with $p_1 = p_2 = 2$.

We claim that $\forall f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$\mathcal{F}_{ext} = \hat{f}$: by lemma 7.20 let $(\varphi_k)_{k \geq 1}$ be a sequence in $C_{00}^\infty(\mathbb{R}^n)$ with $\varphi_k \to f$ in L^1 and L^2. Then

$$\mathcal{F}_{ext}(f) = \lim_{k \to \infty} \mathcal{F}(\varphi_k) \quad \text{in } L^2$$

and in particular there is a subsequence $(k_\ell)_{\ell \geq 1}$ such that $\mathcal{F}_{ext}(f)(x) = \lim_{\ell \to \infty} \mathcal{F}(\varphi_{k_\ell})(x)$

$$= \lim_{\ell \to \infty} \hat{\varphi}_{k_\ell}(x)$$

for almost every $x \in \mathbb{R}^n$.

On the other hand, since $\varphi_k \to f$ in L^1 we have by Prop. 7.2 that $\hat{\varphi}_k \to \hat{f}$ uniformly. This shows that $\mathcal{F}_{ext}(f) = \hat{f}$.

It remains to show the surjectivity of $\mathcal{F}_{ext}$.

~~For any $f \in C_{00}^\infty(\mathbb{R}^n)$ we have~~

For $f \in C_{00}^{\infty}(\mathbb{R}^n)$ we have

$$f = \mathcal{F}^{-1}\mathcal{F}(f).$$

Which we rewrite as:

$$f(x) = \int_{\mathbb{R}^n} \mathcal{F}(f)(\xi) e^{i\langle x,\xi\rangle} \, dm(\xi)$$

$$= \int_{\mathbb{R}^n} \mathcal{F}(f)(-\xi) e^{-i\langle x,\xi\rangle} \, dm(\xi)$$

$$= \mathcal{F}(g)(x)$$

Where $g(\xi) = \mathcal{F}(f)(-\xi)$. Now by lemma 7.5(3) we have $\mathcal{F}(f) \in L^1 \cap L^2$ and hence $g \in L^1 \cap L^2$, which shows that $C_{00}^{\infty}(\mathbb{R}^n) \subset \mathcal{F}_{ext}(L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n))$.

But $\mathcal{F}_{ext}(L^2(\mathbb{R}^n))$ is a closed subspace of $L^2(\mathbb{R}^n)$; it contains $C_{00}^{\infty}(\mathbb{R}^n)$, hence

$$\mathcal{F}_{ext}(L^2(\mathbb{R}^n)) = L^2(\mathbb{R}^n). \quad \blacksquare$$