7.2. Convolution.

This section is devoted to establishing various approximation results, a typical example being lemma 7.20 previously used in Plancherel's theorem. The main tool is convolution which we proceed to define.

Def. 7.14. Let $f_1, f_2 : \mathbb{R}^n \longrightarrow \mathbb{C}$ measurable functions and $x \in \mathbb{R}^n$ such that

$$t \longmapsto f_1(x-t) f_2(t)$$

is in $L^1(\mathbb{R}^n)$. Then we define

$$f_1 * f_2(x) = \int_{\mathbb{R}^n} f_1(x-t) f_2(t) \, dm(t).$$

We recall the following result from Analysis IV [see Iacobelli, Thm 3.7].

Prop. 7.15 Let $1 \leq p \leq +\infty$, $f_1 \in L^1(\mathbb{R}^n)$ and $f_2 \in L^p(\mathbb{R}^n)$. Then for almost every $x \in \mathbb{R}^n$, $t \mapsto f_1(x-t) f_2(t)$ is in $L^1(\mathbb{R}^n)$ and $\| f_1 * f_2 \|_p \leq \| f_1 \|_1 \| f_2 \|_p$.

If $f_1 \in L^1(\mathbb{R}^n)$, $f_2 \in L^\infty(\mathbb{R}^n)$ then for every $x \in \mathbb{R}^n$, $t \mapsto f_1(x-t) f_2(t)$ is in $L^1(\mathbb{R}^n)$ and $f_1 * f_2 \in C^b(\mathbb{R}^n)$.

Cor. 7.15.A If $f_1, f_2 \in L^1(\mathbb{R}^n)$ then $f_1 * f_2 \in L^1(\mathbb{R}^n)$, $\| f_1 * f_2 \|_1 \leq \| f_1 \|_1 \| f_2 \|_1$ and $\widehat{(f_1 * f_2)} = \hat{f}_1 \cdot \hat{f}_2$.

$L^1(\mathbb{R}^n)$ is a fundamental example of (commutative) Banach algebra. This algebra however has no identity as is easy to see using that $\{\hat{f} : f \in L^1(\mathbb{R}^n)\}$ is norm dense in $C_0(\mathbb{R}^n)$. It has, as we will see, approximate identities, that is a sequence $\delta_\varepsilon \in C^\infty_{00}(\mathbb{R}^n)$ $\varepsilon \searrow 0$ such that $\delta_\varepsilon * f \to f$ in L^1, $f \in L^1(\mathbb{R}^n)$ and $\delta_\varepsilon * f \in C^\infty(\mathbb{R}^n)$.

The latter property comes from the following important smoothing property of convolution:

<u>Prop. 7.16.</u> If $f_1 \in C^\infty_{00}(\mathbb{R}^n)$ and $f_2 \in L^p(\mathbb{R}^n)$ then $f_1 * f_2 \in C^\infty(\mathbb{R}^n)$ and

$$\frac{\partial}{\partial x_i}(f_1 * f_2) = \frac{\partial f_1}{\partial x_i} * f_2 \quad 1 \leq i \leq n.$$

Here and in the sequel we will need the following lemma which is a verification using Fubini's thm.

<u>Lemma 7.17</u> Let $(Y, \mathcal{F}, \mu)$ be a σ-finite measure space, $D \subset \mathbb{R}^n$ an open subset and

$$f : D \times Y \longrightarrow \mathbb{C}$$

a measurable function such that:

(1) $x \mapsto f(x,y)$ is in $C^1(D)$ $\forall y \in Y$.

(2) $\forall x \in D$, $1 \leq i \leq n$, $y \mapsto f(x,y)$ and $y \mapsto \frac{\partial f}{\partial x_i}(x,y)$ are in $L^1(Y,\mu)$.

Define
$$F(x) := \int_Y f(x,y)\, d\mu(y)$$

$$G_i(x) = \int_Y \frac{\partial f}{\partial x_i}(x,y)\, d\mu(y).$$

If $G_i \in C(D)$ $1 \leq i \leq n$ then $F \in C^1(D)$ and $\frac{\partial F}{\partial x_i} = G_i$.

<u>Proof</u> (sketch). Let $x \in D$, $v \in \mathbb{R}^n$ and $\varepsilon > 0$ with $x + tv \in D$ $\forall\ |t| \leq \varepsilon$. Then

$$f(x+tv, y) - f(x,y) = t \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x,y)\, v_i\ +$$

$$+ \int_0^t \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}(x+sv, y) - \frac{\partial f}{\partial x_i}(x,y) \right) v_i \, ds$$

Integrating over y and using Fubini we get

$$F(x+tv) - F(x) = t \sum_{i=1}^{n} G_i(x)\, v_i\ +$$

$$+ \int_0^t \sum_{i=1}^{n} \left(G_i(x+sv) - G_i(x) \right) v_i \, ds$$

Now use the continuity of G_i to conclude that

$$\lim_{t \to 0} \frac{1}{t} \int_0^t \left(G_i(x+sv) - G_i(x) \right) ds = 0 .$$

Hence $F \in C^1(D)$ and $\dfrac{\partial F}{\partial x_i} = G_i$.

$\boxed{\beta}$

Proof of Prop. 7.16.

Let $r_1 > 0$ with $\operatorname{supp}(f_1) \subset B_{<r_1}(0)$

and let $D = B_{<r}(0)$ with r arbitrary.

If $\chi_{r+r_1} = \chi_{B_{<r+r_1}(0)}$ then $\forall x \in D$

$$f_1 * f_2(x) = \int_{\mathbb{R}^n} f_1(x-y) f_2(y)\, dm(y)$$

$$= \int_{\mathbb{R}^n} f_1(x-y)\, \chi_{r+r_1}(y) f_2(y)\, dm(y).$$

Now consider $f : D \times \mathbb{R}^n \longrightarrow \mathbb{C}$

$$(x, y) \longmapsto f_1(x-y)\, \chi_{r+r_1}(y) f_2(y)$$

The point is now that since $f_2 \in L^p(\mathbb{R}^n)$

$\chi_{r+r_1} \cdot f_2 \in L^1(\mathbb{R}^n)$. Then the first and

second conditions of lemma 7.17 are satisfied.

Next we have:

$$G_i(x) = \int_{\mathbb{R}^n} \frac{\partial f_1}{\partial x_i}(x-y)\, \chi_{r+r_1}(y)\, f_2(y)\, dm(y)$$

$$= \frac{\partial f_1}{\partial x_i} * \left(\chi_{r+r_1} \cdot f_2\right)(x)$$

which is continuous by Prop. 7.15.

Now apply Lemma 7.17. ∎

Now we turn to the construction of approximate identity.

Let $\delta \in C_{00}^{\infty}(\mathbb{R}^n)$, $r>0$, satisfy

(7.18) $\displaystyle\int_{\mathbb{R}^n} \delta(y)\, dm(y) = 1$ and $\operatorname{supp} \delta \subset B_{<r}(0)$.

$\delta \geq 0$.

Define

(7.19) $\displaystyle\delta_\varepsilon(y) = \frac{1}{\varepsilon^n}\, \delta\!\left(\frac{y}{\varepsilon}\right), \quad \varepsilon > 0.$

Lemma: Assume $\varphi_1; \varphi_2 \in L^1(\mathbb{R}^n)$ and $\operatorname{supp} \varphi_1 \subset K_1$, $\operatorname{supp} \varphi_2 \subset K_2$. With K_1, K_2 ~~say closed~~ compact. Then

$$\operatorname{supp}(\varphi_1 * \varphi_2) \subset K_1 + K_2$$

which is compact.

Proof: $$\varphi_1 * \varphi_2(x) = \int_{\mathbb{R}^n} \varphi_1(x-y)\, \varphi_2(y)\, dm(y).$$

$$= \int_{K_2} \varphi_1(x-y)\, \varphi_2(y)\, dm(y).$$

If now $x \notin K_1 + K_2$, then $x - y \notin K_1$ $\forall y \in K_2$. So for $x \notin K_1 + K_2$

$$\varphi_1(x-y)\, \varphi_2(y) = 0 \quad \forall y \in K_2.$$

$\boxed{\xi}$

Then we have:

(7.20) $\int_{\mathbb{R}^n} \delta_\varepsilon(y)\, dm(y) = 1$, $\operatorname{supp} \delta_\varepsilon \subset B_{\leq \varepsilon \cdot r}(0)$.

Prop. 7.18

(1) If $f \in C(\mathbb{R}^n)$ then $\lim_{\varepsilon \to 0} \delta_\varepsilon * f = f$

uniformly on compact sets.

(2) If $f \in L^p(\mathbb{R}^n)$, $1 \leq p < +\infty$ then

$$\lim_{\varepsilon \to 0} \| \delta_\varepsilon * f - f \|_p = 0.$$

⟹

Proof:

(1) $\delta_\varepsilon * f(x) = \int_{\mathbb{R}^n} \delta_\varepsilon(y) f(x-y)\, dm(y)$ (change of var.)

Hence, taking into account that

$$\int_{\mathbb{R}^n} \delta_\varepsilon(y)\, dm(y) = 1$$

We get

$$\delta_\varepsilon * f(x) - f(x) = \int_{\mathbb{R}^n} \delta_\varepsilon(y)\big(f(x-y) - f(x)\big)\,dm(y)$$

Thus

$$|\delta_\varepsilon * f(x) - f(x)| \leq \sup_{y \in B_{<\varepsilon \cdot r}(0)} |f(x-y) - f(x)|$$

which leads to (1) taking into account that f is uniformly continuous on compact sets.

(2) Let $f \in L^p(\mathbb{R}^n)$, $0 < \varepsilon \leq 1$ and $\varphi \in C_{00}(\mathbb{R}^n)$ with $\|f - \varphi\|_p < \varepsilon$. (recall $p < +\infty$).

Then:

$$\|\delta_\varepsilon * f - f\|_p \leq \|\delta_\varepsilon * f - \delta_\varepsilon * \varphi\|_p + \|\delta_\varepsilon * \varphi - \varphi\|_p$$
$$+ \|\varphi - f\|_p$$

Now $\|\delta_\varepsilon * f - \delta_\varepsilon * \varphi\|_p = \|\delta_\varepsilon * (f - \varphi)\|_p$

$$\leq \|\delta_\varepsilon\|_1 \cdot \|f - \varphi\|_p \quad \text{(Prop. 7.15)}$$

$$= \|f - \varphi\|_p .$$

Hence $\| \delta_\varepsilon * f - f\|_p < 2\cdot\varepsilon + \|\delta_\varepsilon * \varphi - \varphi\|_p$.

Now

$$\operatorname{supp}(\delta_\varepsilon * \varphi) \overset{\text{lemma above}}{\subset} B_{<\varepsilon\cdot r}(0) + \operatorname{supp}\varphi$$

$$\subset B_{<r}(0) + \operatorname{supp}\varphi =: K.$$

Thus

$$\|\delta_\varepsilon * \varphi - \varphi\|_p^p = \int_K |\delta_\varepsilon * \varphi(y) - \varphi(y)|^p \, dm(y)$$

$$\leq \sup_{y\in K} |\delta_\varepsilon * \varphi(y) - \varphi(y)|^p \, m(K)$$

$$\downarrow \quad \varepsilon \to 0 \quad \text{by (1).}$$

$$0$$

$\boxed{\}}$

Here is an important Corollary:

<u>Cor. 7.19</u>: Let $\Omega \subset \mathbb{R}^n$ be open and $1 \leq p < +\infty$. Then $C_{00}^\infty(\Omega)$ is dense in $L^p(\Omega)$.

<u>Lemma</u>: $K \subset \Omega$ K compact. Then $\exists\, \varepsilon > 0$ s.t. $K + B_{<\varepsilon}(0) \subset \Omega$.

<u>Proof</u> Since Ω is locally compact and $m|_{\Omega}$ is Borel regular we know that

$$C_{00}(\Omega) \text{ is dense in } L^p(\Omega).$$

Let $f \in C_{00}(\Omega)$. Since $supp(f) \subset \Omega$, Ω is open and $supp(f)$ is compact there is $\varepsilon_0 > 0$ with $supp(f) + B_{<\varepsilon_0 r}(0) \subset \Omega$.

Then $\forall \varepsilon < \varepsilon_0$, $supp(\delta_\varepsilon * f) \subset \Omega$, $\delta_\varepsilon * f \in C^\infty_{00}(\Omega)$ (Prop. 7.16) and

$$\lim_{\varepsilon \to 0} \| \delta_\varepsilon * f - f \|_p = 0 \quad \text{by Prop 7.18. (2).}$$

$\square$

Here is an application that's used in Plancherel:

<u>Lemma 7.20</u> Let $1 \leq p_1, p_2 < +\infty$, and $f \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$. Then $\exists (\varphi_k)_{k \geq 1}$ in $C^\infty_{00}(\mathbb{R}^n)$ with $\varphi_k \longrightarrow f$ in L^{p_1} and L^{p_2}.

<u>Proof:</u> Let $\ell \in \mathbb{N}$, $\ell \geq 1$, $\chi_\ell = \chi_{B(0)_{<\ell}}$.

Then $f \cdot \chi_\ell \longrightarrow f$ in L^{p_1} and L^{p_2}.

[give argument].

Next, $f \cdot \chi_\ell \in L^1(\mathbb{R}^n)$ and has compact ~~support~~ support. Hence (lemma above)

$$\delta_{1/k} * f \cdot \chi_\ell$$

has compact support and is in $C^\infty(\mathbb{R}^n)$ (Prop. 7.16).

Thus $\delta_{1/k} * f \cdot \chi_\ell \in C^\infty_{00}(\mathbb{R}^n)$.

Now by Prop. 7.18, $\delta_{1/k} * f \cdot \chi_\ell \xrightarrow{k \to \infty} f \cdot \chi_\ell$ in L^{p_1} and L^{p_2}. Thus by a diagonal argument $\exists\, (s_m)_{m \geq 1}$ of integers such that

$$\delta_{1/s_m} * f \cdot \chi_{s_m} \longrightarrow f \text{ in } L^{p_1} \text{ and } L^{p_2}.$$

$\blacksquare$

7.3. Weak derivatives.

Let $\Omega \subset \mathbb{R}^n$ be open. Recall that for $f \in C^\infty(\Omega)$ and $\varphi \in C^\infty_{00}(\Omega)$, integration by parts gives:

$$\int_\Omega \cancel{f}\, D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega (D^\alpha f) \cdot \varphi$$

This motivates the following def.:

Def. 7.21 Let $f \in L^1_{loc}(\Omega)$. ~~The~~ $\cancel{f}$

A function $h \in L^1_{loc}(\Omega)$ is a weak α-th partial derivative of f if

$$\int_\Omega h \cdot \varphi = (-1)^{|\alpha|} \int_\Omega f \cdot D^\alpha \varphi \quad \forall \varphi \in C^\infty_{00}(\Omega)$$

Since φ and $D^\alpha \varphi$ are compactly supported, these integrals make sense.

First we want to show that ~~if~~ if it exists, such a weak derivative is unique. This follows from

<u>Lemma 7.22</u> With $\Omega \subset \mathbb{R}^n$ open as above and $h \in L^1_{loc}(\Omega)$. If

$$\int h(x)\,\varphi(x)\,dm(x) = 0 \quad \forall \varphi \in C^\infty_{00}(\Omega)$$

then $h = 0$ almost everywhere.

<u>Proof:</u> Let $x \in \Omega$, $r > 0$, $\varepsilon > 0$ with $\underset{\leq r+\varepsilon}{B}(x) \subset \Omega$. Set $\chi = \chi_{\underset{\leq r}{B}(x)}$.

Observe that if we had $h \in L^2_{loc}(\Omega)$ then $\chi \cdot h \in L^2\left(B_{\leq r}(x)\right)$ and $\forall$ $\varphi \in C^\infty_{00}(B_{<r}(x))$ we have

$$\int \chi \cdot h \cdot \varphi = 0 \quad \text{which shows}$$

$C^\infty_{oo}(B_{<r}(x))$ is dense in $L^2(B_{<r}(x))$ (Cor 7.19)

this would imply $h \cdot \chi = 0$ (a.e) and since

$B_{<r}(x) \subset \Omega$ is arbitrary, $h = 0$ (a.e.).

So we modify this idea: $\chi \cdot h \in L^1(\mathbb{R}^n)$,

and we will consider $\delta_\varepsilon * \chi \cdot h$ ~~which~~

where $\operatorname{supp}(\delta) \subset B_{<1}(0)$ and $\delta(t) = \delta(-t)$
$\forall t \in \mathbb{R}^n$.

Now $\forall \varphi \in C^\infty_{oo}(\mathbb{R}^n)$,

$$\int (\delta_\varepsilon * \chi \cdot h) \cdot \varphi = \int \chi h (\delta_\varepsilon * \varphi)$$

[Computation]. If now $\operatorname{supp} \varphi \subset B_{<r}(x)$

then $\operatorname{supp}(\delta_\varepsilon * \varphi) \subset B_{<r+\varepsilon}(x) \subset \Omega$.

Hence $\int (\delta_\varepsilon * \chi \cdot h) \cdot \varphi = \int \chi \cdot h \, (\delta_\varepsilon * \varphi) = 0$

for such φ's. Next by Prop. 7.16,

$$\delta_\varepsilon * \chi \cdot h \in C^\infty(\mathbb{R}^n)$$

and hence its restriction to $B_{<r}(x)$ is L^2.

Since $C^\infty_{00}(B_{<r}(x))$ is dense in $L^2(B_{<r}(x))$

we get $\delta_\varepsilon * \chi \cdot h = 0$. in $B_{<r}(x)$. Since

$$\delta_\varepsilon * \chi \cdot h \longrightarrow \chi \cdot h \quad \text{in} \quad L^1(\mathbb{R}^n)$$

we get $\quad \chi \cdot h = 0 \quad$ a.e. in $B_{<r}(x)$

and hence $\quad h = 0 \quad$ a.e. in $B_{<r}(x)$.

$\boxed{\text{B}}$

Notation 7.23 If h is the weak α-th partial derivative of f we'll write $h = D_w^\alpha f$.

In particular it follows from lemma 7.22 that if $f \in C^\infty(\Omega)$, $D_w^\alpha f = D^\alpha f$.

Example 7.24. Let $r > 0$ and $f(t) = |t|^r$.

Then $f \in L^1_{loc}(\mathbb{R})$ and

$$\left(\frac{d}{dt}\right)_w f(t) = r \operatorname{sgn}(t) |t|^{r-1}$$

which is verified directly by integration by parts.

Now we are in a position to define the Sobolev spaces $W^{p,k}(\Omega)$:

<u>Def. 7.25</u> $\Omega \subset \mathbb{R}^n$ open subset, $p \geq 1$, $k \in \mathbb{N}$:

$$W^{p,k}(\Omega) = \{ f \in L^1_{loc}(\Omega) : D_w^\alpha f \text{ exists } \forall\, |\alpha| \leq k \text{ and } D_w^\alpha f \in L^p(\Omega) \}.$$

On $W^{p,k}(\Omega)$ we put the norm,

$$\|f\|_{p,k} = \sum_{|\alpha| \leq k} \|D_w^\alpha f\|_p .$$

Then we have

<u>Prop. 7.26</u> $W^{p,k}(\Omega)$ is a Banach space.

<u>Proof</u>: Let $(f_\ell)_{\ell \geq 1}$ be a Cauchy sequence in $W^{p,k}(\Omega)$.

Then $\forall\ |\alpha| \leq k$, $(D_w^\alpha f_\ell)_{\ell \geq 1}$ is a Cauchy sequence in $L^p(\Omega)$, hence there is a limit $f^\alpha \in L^p(\Omega)$; in particular $f_\ell \to f$.

Now by definition we have $\forall\ \varphi \in C_{00}^\infty(\Omega)$

$$\int f_\ell D^\alpha \varphi = (-1)^{|\alpha|} \int (D_w^\alpha f_\ell) \cdot \varphi .$$

Now $\varphi \in L^q(\Omega)$, $D_w^\alpha f_\ell \in L^p(\Omega)$ and L^p-converges to f^α, hence

$$\lim_\ell \int f_\ell D^\alpha \varphi = (-1)^{|\alpha|} \int f^\alpha \cdot \varphi$$

Next $f_\ell \longrightarrow f$ in L^p and hence

$$\int f D^\alpha \varphi = (-1)^{|\alpha|} \int f^\alpha \varphi$$

which implies $D_w^\alpha f = f^\alpha$. $\boxed{}$

<u>Remark 7.27</u> $W^{2,k}(\Omega)$ is a Hilbert space.

Indeed for $f_1, f_2 \in W^{2,k}(\Omega)$,

$$\langle f_1, f_2 \rangle = \sum_{|\alpha| \leq k} \langle D_w^\alpha f_1, D_w^\alpha f_2 \rangle$$

is well defined, and the norm

$$\| f \| = \left(\sum_{|\alpha| \leq k} \| D_w^\alpha f \|_2^2 \right)^{1/2}$$

is equivalent to $\| \ \|_{2,k}$.

Let $C^\infty_{p,k}(\Omega) = \{ f \in C^\infty(\Omega) :$

$$\| D^\alpha f \|_p < +\infty \quad \forall |\alpha| \leq k \}$$

Then $C^\infty_{p,k}(\Omega) \subset W^{p,k}(\Omega)$ and it is a fact that the former is dense in the latter. In the sequel we will show this for $\Omega = \mathbb{R}^n$.

To this end we collect some simple facts about weak derivatives.

Lemma 7.28

(a) If $f \in W^{p,k}(\Omega)$ and $|\alpha|+|\beta| \leq k$ then $D_w^\alpha(D_w^\beta f) = D_w^{\alpha+\beta} f$. (addition of multiind.).

(b) If $f \in W^{p,k}(\Omega)$ and $\varphi \in C_{00}^\infty(\Omega)$ then $\varphi \cdot f \in W^{p,k}(\Omega)$.

(c) If $\varphi \in C_{00}^\infty(\mathbb{R}^n)$, $f \in W^{p,k}(\mathbb{R}^n)$ then $\varphi * f \in C_{p,k}^\infty(\mathbb{R}^n)$ and

$$D^\alpha(\varphi * f) = \varphi * D_w^\alpha f \quad \forall |\alpha| \leq k.$$

Proof (a) We have $\forall \varphi \in C_{00}^\infty(\Omega)$

$$\int D_w^\alpha(D_w^\beta f)\varphi = (-1)^{|\alpha|}\int (D_w^\beta f) D^\alpha \varphi$$

$$= (-1)^{|\alpha|+|\beta|}\int f D^\beta D^\alpha \varphi$$

But $D^\beta D^\alpha \varphi = D^{\beta+\alpha}\varphi$, hence

$$= (-1)^{|\alpha+\beta|}\int f\, D^{\beta+\alpha}\varphi$$

$$= \int (D_w^{\beta+\alpha} f)\,\varphi$$

which by uniqueness (lemma 7.22) implies

$$D_w^\alpha (D_w^\beta f) = D_w^{\alpha+\beta} f.$$

(b) We may assume $k \geq 1$: let $\psi \in C_{00}^\infty(\Omega)$,

$$\int (\varphi \cdot f)\,\partial_j \psi = \int f\, \varphi \cdot \partial_j \psi = \int f\,[\partial_j(\varphi \cdot \psi) - \psi\,\partial_j \varphi]$$

$$= \int f\,\partial_j(\varphi\cdot\psi) - \int f\,\psi\,\partial_j\varphi$$

$$= -\int (\partial_j^w f)\,\varphi\cdot\psi - \int f\,\partial_j\varphi\cdot\psi$$

$$= -\int [(\partial_j^w f)\cdot\varphi + f\,\partial_j\varphi]\cdot\psi$$

Since $\partial_j^{\psi} f \in L^p(\Omega)$ and $f \in L^p(\Omega)$

so is $(\partial_j^{\psi} f)\varphi + f\cdot(\partial_j \varphi)$. Hence

$\partial_j^{w}(\varphi\cdot f)$ exists, is in $L^p(\Omega)$ and

$$\partial_j^{w}(\varphi\cdot f) = \varphi\cdot \partial_j^{w} f + f\,\partial_j \varphi.$$

One completes the proof by recurrence on $|\alpha|$

using $$D^{\alpha}(\varphi\cdot\psi) = \sum_{0\leq\beta\leq\alpha} \binom{\alpha}{\beta} D^{\beta}\varphi\, D^{\alpha-\beta}\psi.$$

(c) We know that by Prop. 7.16:

$$\varphi * D_w^{\alpha} f \in C^{\infty}(\mathbb{R}^n) \qquad |\alpha| \leq k$$

and by Prop. 7.15,

$$\varphi * D_w^{\alpha} f \in L^p(\mathbb{R}^n), \qquad |\alpha| \leq k.$$

~~Hence~~. Thus it suffices to show that

$$D^{\alpha}(\varphi * f) = \varphi * D_w^{\alpha} f.$$

We have $\forall \psi \in C_{00}^{\infty}(\mathbb{R}^n)$:

$$\int (\varphi * D_W^\alpha f)\,\psi$$

$$= \int (D_W^\alpha f)\ \check{\varphi} * \psi \qquad \text{where } \varphi^\vee(x) = \varphi(-x).$$

$$= (-1)^{|\alpha|} \int f \ \underbrace{D^\alpha(\varphi^\vee * \psi)}_{\check{\varphi} * D^\alpha \psi}$$

$$= (-1)^{|\alpha|} \int (\varphi * f)\, D^\alpha \psi$$

$$= \int D^\alpha(\varphi * f) \cdot \psi$$

Which implies $D^\alpha(\varphi * f) = \varphi * D_W^\alpha f$.

$\boxed{\xi}$

With this we can now prove

<u>Prop. 7.29</u> $1 \leq p < +\infty$, then $C^{\infty}_{p,k}(\mathbb{R}^n)$ is dense in $W^{p,k}(\mathbb{R}^n)$.

<u>Proof</u>: Fix (δ_ε) approximate identity and let $f \in W^{p,k}(\mathbb{R}^n)$. Then by lemma 7.28 c) we know that $\delta_\varepsilon * f \in C^{\infty}_{p,k}(\mathbb{R}^n)$ and

$$D^\alpha(\delta_\varepsilon * f) = \delta_\varepsilon * D^\alpha_w f \,, \quad |\alpha| \leq k.$$

By Prop. 7.18 (2) we have $\forall\ |\alpha| \leq k$

$$\delta_\varepsilon * D^\alpha_w f \longrightarrow D^\alpha_w f \quad \text{in } L^p(\mathbb{R}^n).$$

Hence $D^\alpha(\delta_\varepsilon * f) \longrightarrow D^\alpha_w f$ in $L^p(\mathbb{R}^n)$

and Hence $\| \delta_\varepsilon * f - f \|_{p,k} \longrightarrow 0$. $\boxed{\Sigma}$

7.4. The Sobolev embedding theorems.

The aim of this section is to prove

Thm 7.30 (Sobolev) If $f \in W^{2,k}(\mathbb{R}^n)$ and $k > r + \frac{n}{2}$ then $f \in C^r_b(\mathbb{R}^n)$. Moreover the inclusion $\mathcal{E} \; W^{2,k}(\mathbb{R}^n) \to C^r_b(\mathbb{R}^n)$ is bounded. □

Remark 7.31. More precisely, f coincides a.e. with a bounded C^r-function. The norm on $C^r_b(\mathbb{R}^n)$ is

$$\|f\| = \sum_{|\alpha| \le r} \|D^\alpha f\|_b .$$

Interestingly, Fourier transform and in particular Plancherel's theorem will be crucial, while these objects don't appear in the statement.

We will need 3 lemmas. The first is an extension to $W^{2,k}$ of Prop 7.5. (2).

<u>Lemma 7.32.</u> $f \in W^{2,k}(\mathbb{R}^n)$. Then $\forall\, |\alpha| \leq k$: $\widehat{(D^\alpha_w f)}(\xi) = i^{|\alpha|}\xi^\alpha \hat{f}(\xi)$.

Here $\wedge : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the L^2-Fourier transform. [[In particular $\xi^\alpha \hat{f} \in L^2$ $\forall\, |\alpha| \leq k$.]]

<u>Proof:</u>

By induction on $k \geq 1$; here we show $k=1$. We have: $\partial_j f \in L^2$; then $\forall \varphi \in C^\infty_{sol}$

$$\langle \widehat{\partial^w_j f}, \hat{\varphi} \rangle \overset{Pl.}{=} \langle \partial^w_j f, \varphi \rangle$$

$$= -\langle f, \partial_j \varphi \rangle$$

$$\overset{Pl}{=} -\langle \hat{f}, \widehat{\partial_j \varphi} \rangle$$

But by Prop. 7.5 (2): $\widehat{\partial_j \varphi}(\xi) = i \cdot \xi_j \hat{\varphi}(\xi)$.

Thus $-\langle \hat{f}, \widehat{\partial_j \varphi} \rangle = \int \xi_j \hat{f}(\xi) \overline{\hat{\varphi}(\xi)}$

and since $\{ \hat{\varphi} : \varphi \in C^\infty_{00}(\mathbb{R}^n) \}$ is dense in $L^2(\mathbb{R}^n)$ (follows from Cor. 7.19 and Plancherel) we get:

$$(\widehat{\partial^w_j f})(\xi) = i \cdot \xi_j \hat{f}(\xi).$$

$\boxed{\text{//}}$

<u>Remark</u>: The idea of Sobolev's thm. is to exploit Lemma 7.32 and Plancherel: indeed if $D^\alpha_w f \in L^2(\mathbb{R}^n)$, then $\xi^\alpha \hat{f} \in L^2$ and if k is large enough, Fourier inversion gives a sup bound on f and some derivatives:

here the hypothesis $k > r + \frac{n}{2}$ is crucial.

For the sake of applications we formulate the next lemma in terms of inverse F.T.

Recall: for $h \in L^1(\mathbb{R}^n)$,

$$\tilde{h}(x) = \int_{\mathbb{R}^n} h(\xi) e^{i\langle x, \xi \rangle} dm(\xi).$$

<u>Lemma 7.33</u> Let $r \in \mathbb{N}$, $h \in L^1(\mathbb{R}^n)$ and assume $\xi^\alpha h \in L^1(\mathbb{R}^n)$ $\forall |\alpha| \le r$. Then $\tilde{h} \in C^r_b(\mathbb{R}^n)$ and

$$D^\alpha \tilde{h}(x) = i^{|\alpha|} \widetilde{(\xi^\alpha h)}(x).$$

<u>Proof</u>: This is again a case where we want to exchange derivative with integration.

<u>$r = 1$.</u> $$\tilde{h}(x) = \int_{\mathbb{R}^n} h(\xi) e^{i\langle x, \xi \rangle} dm(\xi).$$

We want to apply lemma 7.17 to

$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, $f(x, \xi) = h(\xi) e^{i\langle x, \xi \rangle}$;

then indeed

(1) $x \to f(x, \xi)$ is in $C^1(\mathbb{R}^n)$

(2) $\xi \to f(x, \xi)$ and $\xi \to \frac{\partial f}{\partial x_j}(x, \xi)$

$\in L^1(\mathbb{R}^n)$, since

$$\frac{\partial f}{\partial x_j}(x, \xi) = h(\xi)\, i \cdot \xi_j\, e^{i\langle x, \xi \rangle}.$$

Moreover: $F(x) = \int_{\mathbb{R}^n} f(x, \xi)\, dm(\xi)$

and $G_j(x) = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x, \xi)\, dm(\xi)$

read resp. $\tilde{h}(x) = \int_{\mathbb{R}^n} h(\xi) e^{i\langle x, \xi \rangle}\, dm(\xi)$

$$G_j(x) = \int_{\mathbb{R}^n} h(\xi)\, i \cdot \xi_j\, e^{i\langle x, \xi \rangle}\, dm(\xi)$$

Thus lemma 7.17 implies $\tilde{h} \in C_b^1(\mathbb{R}^n)$

and $\partial_j \tilde{h} = i\, \widetilde{(\xi_j \cdot h)}$. $\boxed{R}$

Here comes the crucial

<u>Lemma 7.34.</u> Let $r \geq 0$. Assume,

$f \in L^2(\mathbb{R}^n)$ and $\xi^\alpha \hat{f} \in L^1(\mathbb{R}^n)$, $|\alpha| \leq r$.

Then $f \in C_b^r(\mathbb{R}^n)$ and

$$D^\alpha f = i^{|\alpha|}\, \widetilde{(\xi^\alpha \hat{f})}. \quad \forall\, |\alpha| \leq r.$$

In particular $\|D^\alpha f\|_b \leq \|\xi^\alpha \hat{f}\|_1$.

<u>Remark</u>: The case $r = 0$ is already interesting as it extends the Fourier inversion formula to the case, $f \in L^2(\mathbb{R}^n)$, $\hat{f} \in L^1(\mathbb{R}^n)$.

<u>Proof</u>: Apply the preceding lemma to $h = \hat{f}$: then we get $\tilde{h} \in C_b^r(\mathbb{R}^n)$

and $D^\alpha \tilde{h}(x) = i^{|\alpha|}\, \widetilde{(\xi^\alpha h)}(x)$.

But now $f \in L^2(\mathbb{R}^n)$ and hence by Pl.

$\hat{f} \in L^2(\mathbb{R}^n)$, which implies $\tilde{h} = f$.

$\boxed{\xi}$

<u>Proof of Sobolev.</u>

By lemma 7.32 we have $\xi^\alpha \hat{f} \in L^2(\mathbb{R}^n)$ $\forall\, |\alpha| \leq k$ and we want to apply Lemma 7.34, so we must show that $\xi^\alpha \hat{f} \in L^1(\mathbb{R}^n)$ $\forall\, |\alpha| \leq r$ and it is here that the hypothesis $k > r + \frac{n}{2}$ enters.

Split $\quad \xi^\alpha \hat{f} = h_1 \cdot h_{2,\alpha}$ where

$$h_1(\xi) = \left(1 + \|\xi\|^k\right) \hat{f}$$

$$h_{2,\alpha}(\xi) = \frac{\xi^\alpha}{1 + \|\xi\|^k}, \quad \text{this for } |\alpha| \leq r.$$

Then: $\quad \|\xi^\alpha \hat{f}\|_1 \leq \|h_1\|_2 \cdot \|h_{2,\alpha}\|_2 .$

Let's estimate $\|h_1\|_2$ by relating it to the Sobolev norm of f, using Plancherel.

Now

$$\|\xi\| \leq \sum_{j=1}^{n} |\xi_j| \leq n^{1-1/k} \left(\sum_{j=1}^{n} |\xi_j|^k \right)^{1/k}$$

hence $$\|\xi\|^k \leq n^{k-1} \left(\sum_{j=1}^{n} |\xi_j|^k \right).$$

Hence

$$|h_1| \leq n^{k-1} \left(|\hat{f}| + \sum_{j=1}^{n} |\xi_j^k \hat{f}| \right)$$

Hence $$\|h_1\|_2 \leq n^{k-1} \left(\|\hat{f}\|_2 + \sum_{j=1}^{n} \|\xi_j^k \hat{f}\|_2 \right).$$

Which by lemma 7.32 and Plancherel implies

$$\|h_1\|_2 \leq n^{k-1} \left(\|f\|_2 + \sum_{j=1}^{n} \|\partial_j^k f\|_2 \right).$$

$$\lesssim n^{k-1} \|f\|_{2,k}.$$

Next:

$$|h_{2,\alpha}(\xi)| = \frac{|\xi^\alpha|}{1+\|\xi\|^k}$$

Now $|\xi^\alpha| = |\xi_1|^{\alpha_1} \cdots |\xi_n|^{\alpha_n}$ and

$$|\xi_j|^{\alpha_j} \leq \|\xi\|^{\alpha_j}$$

hence $$|h_{2,\alpha}(\xi)| \leq \frac{\|\xi\|^{|\alpha|}}{1+\|\xi\|^k} \leq \frac{1+\|\xi\|^r}{1+\|\xi\|^k}$$

since $|\alpha| \leq r$.

Thus

$$\int_{\mathbb{R}^n} \frac{(1+\|\xi\|^r)^2}{(1+\|\xi\|^k)^2}\, dm(\xi) =$$

$$= c_n \int_0^\infty dt\, t^{n-1} \frac{(1+t^r)^2}{(1+t^k)^2} < +\infty$$

$\iff k > r + \frac{n}{2}$.

Thus let $C(n,r,k)$ this latter constant

Then $\| \xi^{\alpha} \hat{f} \|_{1} \leqslant C(n,r,k)\, n^{k-1} \, \| f \|_{2,k}$

And hence by lemma 7.34:

$$\| D^{\alpha} f \|_{b} \leqslant C(n,k,k)\, n^{k-1} \, \| f \|_{2,k}$$

$$\forall \ |\alpha| \leqslant r.$$

∎