Example 6.19 (Continued.)

Here we give a proof of Thm 2.8 using the MK fixed point theorem.

We consider $G = \mathbb{R}/\mathbb{Z}$ with quotient topology, $\pi : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ the canonical projection.

For $f \in C(\mathbb{R}/\mathbb{Z})$ let $\mathcal{L}(f) = \int_0^1 f(\pi(x))\, d\mathcal{L}^1(x)$

Where $\mathcal{L}^1$ is the Lebesgue measure. If

$$L_t : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$$
$$x \longmapsto x + t$$

then $\mathcal{L}(f \circ L_t) = \mathcal{L}(f) \quad \forall f \in C(\mathbb{R}/\mathbb{Z})$
$\forall t \in \mathbb{R}/\mathbb{Z}$.

<u>Thm. 2.8</u>: There is a mean $m : \ell^\infty(\mathbb{R}/\mathbb{Z}) \to \mathbb{R}$ such that

(1) $\lambda^*(t)\, m = m \quad \forall t$

(2) $m(f) = \mathcal{L}(f) \quad \forall f \in C(\mathbb{R}/\mathbb{Z})$.

Let's start with an observation valid $\forall$ group G : TFAE

(1) $m \in \ell^\infty(G)^*$ is a mean.

(2) $m \in \ell^\infty(G)^*$ satisfies $m(\mathbb{1}_G) = 1$ and $\|m\| \leq 1$.

($\Rightarrow$) has been explained.

($\Leftarrow$) If $0 \leq f \leq 1$ then $0 \leq \mathbb{1} - f \leq 1$. So

$$1 = m(\mathbb{1}) = m(f) + m(1-f) \leq m(f) + 1$$

hence $m(f) \geq 0$.

<u>Proof of Thm 2.18.</u>

The map $i : C(\mathbb{R}/\mathbb{Z}) \hookrightarrow \ell^\infty(\mathbb{R}/\mathbb{Z})$ is linear, norm preserving; its adjoint is the restriction:

$$i^* : \ell^\infty(\mathbb{R}/\mathbb{Z})^* \longrightarrow C(\mathbb{R}/\mathbb{Z})^*$$

$$m \longmapsto m|_{C(\mathbb{R}/\mathbb{Z})}$$

and is weak*-cont. by Prop. 5.24(2). ~~...~~

Let $\mathcal{M}_{\mathcal{L}} = \{ m \in \mathcal{M}(\mathbb{R}/\mathbb{Z}) : m|_{C(\mathbb{R}/\mathbb{Z})} = \mathcal{L} \}$.

Then $\mathcal{M}_{\mathcal{L}}$ is weak*-closed in $\mathcal{M}(\mathbb{R}/\mathbb{Z})$ hence compact; it is also convex and $\lambda(t)^*$-inv.

Let's show $\mathcal{M}_{\mathcal{L}} \neq \phi$: we have

$$\mathcal{L} : C(\mathbb{R}/\mathbb{Z}) \longrightarrow \mathbb{R}$$

satisfies $\mathcal{L}(\mathbb{1}) = 1$ and $|\mathcal{L}(f)| \leq \|f\|_{\infty}$

By Hahn-Banach there is an extension

$$m : \ell^{\infty}(\mathbb{R}/\mathbb{Z}) \longrightarrow \mathbb{R}$$

with $|m(f)| \leq \|f\|_{\infty}$; since $m(\mathbb{1}) = \mathcal{L}(\mathbb{1}) = 1$ we get that $m \in \mathcal{M}_{\mathcal{L}}$. Now apply KM FPT. $\boxtimes$

Example 6.18 (second season).

Cor. 6.23. Let G be an abelian group. Then there exists a mean $m \in \mathfrak{M}(G)$ that is $\lambda^*(g)$-inv. $\forall g \in G$. In particular there is a set function $\mu : \mathcal{P}(G) \longrightarrow [0,1]$

with (1) $\mu(G) = 1$

(2) μ is finitely additive.

(3) $\mu(gE) = \mu(E)$ $\forall g \in G$, $\forall E \in E$.

Clearly if G is infinite, then $\mu(E) = 0$ $\forall E \subset G$ finite.

Def. G is *amenable* if $\exists$ a mean m that is $\lambda^*(g)$-inv. $\forall g \in G$.

The free group $\mathbb{F}(a,b)$ on 2 generators.

Word in $S = \{a, b, a^{-1}, b^{-1}\}$ is a finite sequence

$w = s_1 \cdot s_2 \cdot \dots \cdot s_n$

of elements $s_i \in \{a, b, a^{-1}, b^{-1}\}$.

A word w is reduced if it does not contain aa^{-1}, $a^{-1}a$, bb^{-1}, $b^{-1}b$ as subword.

Examples: $a \cdots a \cdot b \cdot a^{-1} \cdot b^{-1}$ is reduced.

$a \cdots a\, b\, b^{-1}\, a^{-1}$ is not.

Let e be the empty word.

Operation of reduction: if w is not reduced, replace any occurrence of aa^{-1}, $a^{-1}a$, bb^{-1}, $b^{-1}b$ by the empty word, to reach in finitely many steps a reduced word:

$a \cdot a \cdot a\, b\, b^{-1} a^{-1} \to a\,a\,a\,a^{-1} \to a\,a.$

The elements of $F(a,b)$ are the set of reduced words, including e.

The product $W_1 \cdot W_2$ of two elements of $F(a,b)$ is the reduced word obtained after successive reduction of the concatenation of W_1 with W_2:

$a\,a\,b\,b \cdot b^{-1} a^{-1}$: $\quad a\,a\,b\,\underbrace{b\,b^{-1}}\,a^{-1}$

$a\,a\,b\,a^{-1}.$

To ease notation one denotes

$$a^n = \underbrace{a \cdot \ldots \cdot a}_{n \text{ times}} \qquad n \geq 1$$

$$a^n = \underbrace{a^{-1} \cdot \ldots \cdot a^{-1}}_{|n| \text{ times}} \qquad n \leq -1.$$

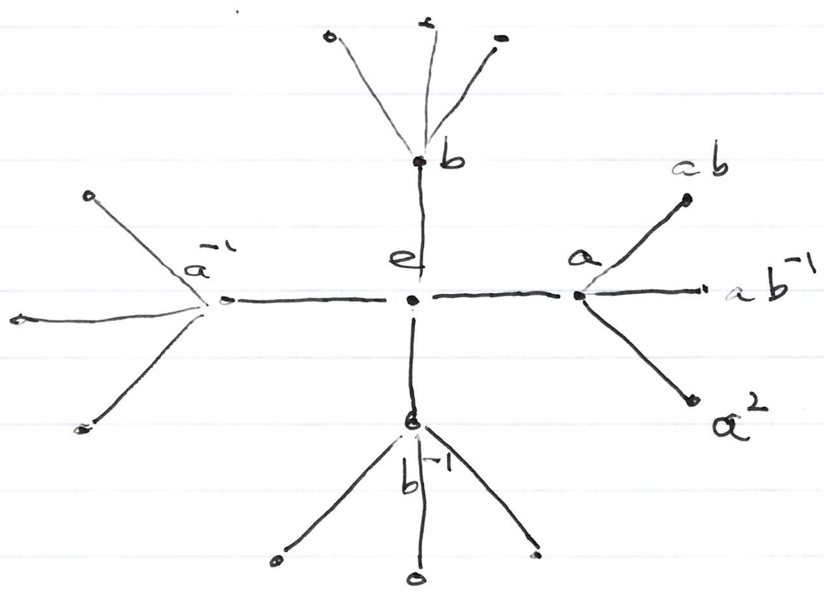
etc. ..

To get a sense how this group looks like we construct its Cayley graph (Gruppenbild):

$\mathrm{Cay}(\mathbb{F}(a,b), S)$:

Vertices: elements of $\mathbb{F}(a,b)$

Edges: two $w_1, w_2 \in \mathbb{F}(a,b)$ are conn. by an edge if $w_2 = w_1 \cdot s$ for some $s \in S$.



Observation: if a reduced word w starts with $s \in S$ and $t \neq s^{-1}$ then $t \cdot w$ is reduced.

For instance: let $W(s) \subset \mathbb{F}(a,b)$ be the subset of reduced words starting with s and $t \neq s^{-1}$. Then:

$$t \cdot W(s) = W(ts) = \text{set of reduced}$$

words starting with ts.

Now assume $\mu : \mathcal{P}(\mathbb{F}(a,b)) \to [0,1]$ is a finitely additive set function with $\mu(G) = 1$ and $\mu(gE) = \mu(E)$ $\forall E \subset \mathbb{F}(a,b)$ $\forall g \in \mathbb{F}(a,b)$.

Then: $W(a) = \{a\} \sqcup W(ab) \sqcup W(ab^{-1}) \sqcup W(a^2)$

Now observe first that $\mu(\{g\}) = 0$ $\forall g \in G$: indeed, $\mu(\{g\}) = \mu(\{e\})$ and $\forall g_1, \dots, g_n$ distinct: $n \cdot \mu(\{e\}) = \mu(\{g_1\}) + \dots + \mu(\{g_n\}) \leq 1$.

Next: from $W(a^2) = a W(a)$

and invariance and additivity we get:

$$\mu(W(a)) = \mu(W(ab)) + \mu(W(ab^{-1})) + \underbrace{\mu(a \cdot W(a))}_{\mu(W(a))}$$

$$\Rightarrow \quad \mu(\underbrace{W(ab)}_{aW(b)}) = 0 \quad , \quad \mu(\underbrace{W(ab^{-1})}_{aW(b^{-1})}) = 0$$

hence $\mu(W(b)) = \mu(W(b^{-1})) = 0$ and

by symm.

$$\mu(W(a)) = \mu(W(a^{-1})) = 0 .$$

But $\mathbb{F}(a,b) = \{e\} \sqcup W(a) \sqcup W(a^{-1}) \sqcup W(b) \sqcup W(b^{-1})$

contradiction.

~~Banach-Tarski~~

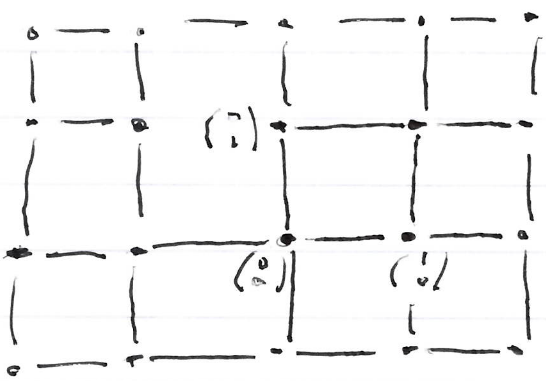
~~We have seen that there are SO(2) invariant finitely additive set functions on S^1 and now we proceed to explain the strategy of~~

<u>Geometric meaning</u> of <u>amenability</u>.

Let G be a group and assume it admits a finite generating set, like $\mathbb{F}(a,b)$ or $\mathbb{Z}^2$. Let $F = F^{-1}$ be one such.

The Cayley graph $Ca(G,F)$ is the graph with vertex set $V = G$ and two vertices g_1, g_2 are connected by an edge if $g_2 = g_1 \cdot s$ for some $s \in F$.

<u>Example</u> $\mathbb{Z}^2$, $F = \left\{ \binom{1}{0}, -\binom{1}{0}, \binom{0}{1}, -\binom{0}{1} \right\}$.



Let $D \subset G$ be any subset of vertices

and $\partial D = \{ v \in D : \exists w \notin D$ such that v and w are connected by an edge $\}$

Then:

Thm. (Fölner) G is amenable $\Leftrightarrow$

$\exists$ sequence $D_n \subset G$, D_n finite

and
$$\frac{|\partial D_n|}{|D_n|} \longrightarrow 0.$$

In other words, G is non-amenable

$\Leftrightarrow \exists\ c > 0$ such that $\forall\ D \subset G$ finite

$$|\partial D| \geq c \cdot |D|.$$

(linear isoperimetric inequality).

Since amenability is a property of the group G, this charact. is independent of the choice of finite generating subset.

Banach – Tarski

Let's now explain the strategy of:

Thm. There is no finitely additive, $SO(3)$ – invariant set function $\mu : \mathcal{P}(S^2) \longrightarrow [0,1]$ with $\mu(S^2) = 1$.

It is based on three facts, two of which are relatively easy to establish.

(1) Let $H < G$; then G acts on the left on G/H, say $\forall g \in G$:

$$L_g : G/H \longrightarrow G/H$$
$$xH \mapsto gxH$$

and we get $\lambda^*(g) : \ell^\infty(G/H)^* \longrightarrow \ell^\infty(G/H)^*$

by $(\lambda^*(g)m)(f) = m(f \circ L_g)$.

Then if H is amenable and there is

a $\lambda^*(g)$-invariant $(g \in G)$ mean on $\ell^\infty(G/H)^*$

then G is amenable.

Indeed if m_H is ~~an~~ invariant mean on $\ell^\infty(H)$

and $m_{G/H}$ a G-invariant one on $\ell^\infty(G/H)$

then define $\forall\ f \in \ell^\infty(G)$,

$$(T_H f)(gH) = m_H(h \longmapsto f(gh)).$$

Then $T_H f \in \ell^\infty(G/H)$ and now

define $m(f) = m_{G/H}(T_H f)$.

(2) Let $H < G$. Assume G is amenable, then H is amenable. Indeed, let $R \subset G$ be a complete set of representatives of $H \backslash G$. Then $G = \bigsqcup_{g \in R} H \cdot g$.

Now define $i : \ell^\infty(H) \to \ell^\infty(G)$

$$f \mapsto i(f)$$

$$i(f)(h \cdot g) = f(h) \quad h \in H, g \in R.$$

If m_G is an invariant mean on $\ell^\infty(G)$ let $m_H(f) := m_G(i(f))$. One verifies easily that m_H is an invariant mean on $\ell^\infty(H)$.

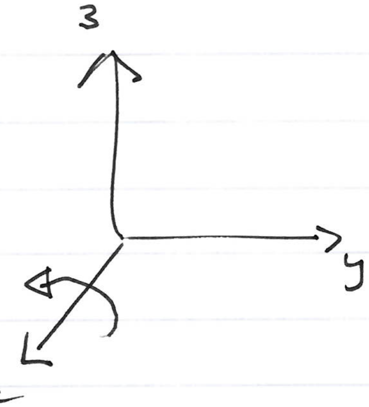
If now $m : \ell^\infty(S^2) \to [0,1]$ were a $SO(3)$-invariant mean, use the orbit map

$$SO(3) \longrightarrow S^2$$
$$g \longmapsto g\begin{pmatrix}0\\0\\1\end{pmatrix}$$

to identify $SO(3)/SO(2) \longrightarrow S^2$.

Then since $SO(2)$ is amenable, fact (1) would imply that $SO(3)$ is amenable.

But here comes



(3) Let a be the rotation about the x-axis with angle θ s.t. $\cos\theta = 1/3$ and b the rotation about the 3-axis with angle θ. Then the subgroup of $SO(3)$ generated by a, b is isomorphic

to $\mathbb{F}(a,b)$.

Fact (2) leads now to a contradiction: if $SO(3)$ were amenable so would $\mathbb{F}(a,b)$. But we showed this is not the case.