Application to Bochner's theorem.

Recall the concepts introduced in 2.2.

We consider $\mathbb{R}/\mathbb{Z}$ with quotient topology and with the addition (mod 1). It is then a compact space and the group operations are continuous.

Let $\mathcal{L}$ be the Lebesgue measure on $\mathbb{R}$ with $\mathcal{L}([0,1]) = 1$ and $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ the canonical projection. We say that $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is measurable if $f \circ \pi$ is $\mathcal{L}$-measurable and we define

$$\int_{\mathbb{R}/\mathbb{Z}} f \, d\mathcal{L}' := \int_0^1 f \circ \pi(x) \, d\mathcal{L}(x).$$

Then $\mathcal{L}' \in M^1(\mathbb{R}/\mathbb{Z})$ is a regular Borel probability measure.

Let $\varphi \in C(\mathbb{R}/\mathbb{Z})$ and let's consider T_K where $K(x,y) := \varphi(x-y)$. Thus for say $f \in L^2(\mathbb{R}/\mathbb{Z}, \mathcal{L}^1)$ ~~...~~

$$T_K f(x) = \int_{\mathbb{R}/\mathbb{Z}} \varphi(x-y) f(y)\, d\mathcal{L}^1(y).$$

Let $\varphi_n(x) := e^{2\pi i n x}$, $n \in \mathbb{Z}$.

Compute

$$T_K \varphi_n(x) = \int_{\mathbb{R}/\mathbb{Z}} \varphi(x-y) e^{2\pi i n y}\, d\mathcal{L}^1(y)$$

$$= \int_{\mathbb{R}/\mathbb{Z}} \varphi(x+y) e^{-2\pi i n y}\, d\mathcal{L}^1(y)$$

$$= e^{2\pi i n x} \int_{\mathbb{R}/\mathbb{Z}} \varphi(y) e^{-2\pi i n y}\, d\mathcal{L}^1(y)$$

<u>Def.</u> The Fourier transform of $\varphi \in L^1(\mathbb{R}/\mathbb{Z}, \mathcal{L}^1)$ is

$$\hat{\varphi}(n) := \int_{\mathbb{R}/\mathbb{Z}} \varphi(y) e^{-2\pi i n y}\, d\mathcal{L}^1(y).$$

<u>Fact</u> $\{\varphi_n : n \in \mathbb{Z}\}$ is an ONB of $L^2(\mathbb{R}/_{\mathbb{Z}}, \mathcal{L}^1)$: the relations $\langle \varphi_n, \varphi_m \rangle = \delta_{n,m}$ are elementary verifications. For completeness observe that

$$\mathcal{A} := \left\{ \sum_{k=-n}^{n} c_k e^{2\pi i k x} : n \geqslant 0, \; c_k \in \mathbb{C} \right\}$$

is a $\mathbb{C}$-algebra in $C(\mathbb{R}/_{\mathbb{Z}})$, invariant under $f \mapsto \bar{f}$, containing $\mathbb{1}$, and separating points. Thus $\mathcal{A}$ is $\| \ \|_b$-dense in $C(\mathbb{R}/_{\mathbb{Z}})$.

If now $f \in L^2(\mathbb{R}/_{\mathbb{Z}}, \mathcal{L}^1)$ with $\langle f, \varphi_n \rangle = 0$ $\forall n \in \mathbb{Z}$ then $\langle f, \varphi \rangle = 0$ $\forall \varphi \in \mathcal{A}$ and hence for all $\varphi \in C(\mathbb{R}/_{\mathbb{Z}})$. This implies $f = 0$.

If $\varphi : \mathbb{R}/_{\mathbb{Z}} \to \mathbb{C}$ is arbitrary then

K is positive semidef. $\Longleftrightarrow$

$$\sum_{i,j=1}^{n} c_i \bar{c}_j \, \varphi(x_i - x_j) \geq 0$$

$\forall n \geq 1$, $\forall x_1, \dots, x_n \in \mathbb{R}/_{\mathbb{Z}}$, $\forall c_1, \dots, c_n \in \mathbb{C}$.

Taking $x_1 = 0$, $x_2 = x$, $c_1 = 1$, $c_2 = c$

we get

$$(1 + |c|^2)\,\varphi(0) + c\,\varphi(x) + \bar{c}\,\varphi(-x) \geq 0 .$$

In particular: $c = 1 \Rightarrow \varphi(0) \geq 0$

and $\varphi(x) + \varphi(-x) \in \mathbb{R}$.

$c = i$: $i(\varphi(x) - \varphi(-x)) \in \mathbb{R}$

$\Longrightarrow$ $\varphi(x) = \overline{\varphi(-x)}$

and $K(x,y) = \overline{K(y,x)}$.

<u>Corollary</u> (Bochner) Let $\varphi \in C(\mathbb{R}/\mathbb{Z})$ be positive semi-definite. Then $\hat{\varphi}(n) \geq 0$, $\forall n \in \mathbb{Z}$ and

$$\varphi(z) = \sum_{n \in \mathbb{Z}} \hat{\varphi}(n) e^{2\pi i n z}$$

with uniform convergence.

<u>Remark</u>: there are cont. $\varphi \in C(\mathbb{R}/\mathbb{Z})$ whose Fourier series diverges on a dense subset of $\mathbb{R}/\mathbb{Z}$!