Thm 4.13 (open mapping thm) Let $V, W$ be Banach spaces and $T : V \to W$ bounded linear. TFAE:

(1) $T$ is surjective.

(2) $T$ is open.

(3) There is a constant $C > 0$ s.t. $\forall w \in W$ the equation $T(v) = w$ has a solution $v \in V$ with $\|v\| \leq C \cdot \|w\|$.

Proof:

(2) $\Rightarrow$ (1) $T(V) \ni 0$ is open; $\forall w \in W$ by continuity of the map $\mathbb{K} \to W$, $\lambda \mapsto \lambda \cdot w$ at $\lambda = 0$, $\exists$ $\mu > 0$ with $\mu \cdot w \in T(V)$ hence $w \in T(V)$ which shows surjectivity.

(3) $\Leftrightarrow$ (2) This uses Exercise 2 in Sheet 2.

Let $T : V \to W$ be a bounded linear operator. Then $\operatorname{Ker} T$ is a closed subspace, $V/\operatorname{Ker} T$ with quotient norm is a Banach space and the canonical projection $\pi : V \to V/_{\operatorname{Ker} T}$ is bounded linear and open. By definition of quotient topology $T$ factors

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi"'] & W \\ V/_{\operatorname{Ker} T} \arrow[ur, "S"'] & \end{tikzcd}$$

and $S$ is linear continuous.

Assume now $T$ open; let $\Omega \subset V/_{\operatorname{Ker} T}$ open, that is, $\pi^{-1}(\Omega) \subset V$ is open. Then $T(\pi^{-1}(\Omega)) = S(\Omega)$ and hence $S(\Omega)$ is open. Now $S$ is bijective open, hence $S^{-1} : W \longrightarrow V/_{\operatorname{Ker} T}$ is continuous.

Thus $\exists\, C > 0$ such that

$$\| S^{-1}(w) \| \leq C \|w\| \quad \forall w \in W.$$

But if $\pi(v) = S^{-1}(w)$ we have

$$\| S^{-1}(w) \| = \inf \{ \|v+n\| : n \in \operatorname{Ker} T \}$$

thus $\exists\, n \in \operatorname{Ker} T$ with

$$\| v + n \| \leq 2 \| S^{-1}(w) \|$$

and thus $\| v + n \| \leq 2 C \|w\|$, which shows (3).

Conversely: (exercise).

(1) $\Longrightarrow$ (2). We have to show that $T$ is open.

Main point is to show that $\exists\, c > 0$

such that $T(B_{<1}(0)) \supset B_{<c}(0)$.

Indeed this implies that $\forall r > 0$,

$T(B_{<r}(0))$ is a neighborhood of $0$ and

hence $T(B_{<r}(v))$ is a neighb. of $T(v)$

$\forall v \in V$, $\forall r > 0$.

If now $\Omega \subset V$ is open, $\forall v \in \Omega$ let $r_v > 0$ with $B_{<r_v}(v) \subset \Omega$. Then $T(\Omega) \supset T(B_{<r_v}(v))$ and hence $T(\Omega)$ is a neighborhood of each of its points, hence open.

Now we come to the proof of the main point. The proof proceeds in several steps.

<u>Step 1</u>. $\exists\, \varepsilon > 0$ with $\overline{T(B_{<1}(0))} \supset B_{<\varepsilon}(0)$.

We have $\bigcup_{n \geq 1} \overline{T(B_{<n}(0))} = W$ and since $W$ is Banach, lemma 4.7 implies that $\exists\, n_0 \geq 1$ with $\overset{\circ}{\overline{T(B_{<n_0}(0))}} \neq \emptyset$. But rescaling by $1/n_0$ is a homeo of $W$, hence

$$\overset{\circ}{\overline{T(B_{<1}(0))}} \neq \emptyset.$$

Thus $\exists\, x \in W$ and $\varepsilon > 0$ so that

$$\overline{T(B_{<1}(0))} \supset B_{<\varepsilon}(x).$$

Now observe that if $w \in \overline{T(B_{<1}(0))}$ then $-w \in \overline{T(B_{<1}(0))}$, and if $w_1, w_2 \in \overline{T(B_{<1}(0))}$ then $\frac{w_1 + w_2}{2} \in \overline{T(B_{<1}(0))}$. Thus

$$B_{<\varepsilon}(-x) = -B_{<\varepsilon}(x) \subset \overline{T(B_{<1}(0))} \text{ and}$$

hence $\forall\, w \in B_{<\varepsilon}(0)$,

$$w = \frac{1}{2}(-x+w) + \frac{1}{2}(x+w) \in \overline{T(B_{<1}(0))}.$$

Which shows step 1. This used that $W$ is Banach.

<u>Step 2</u>. Let $\varepsilon = 2 \cdot c$ with $\overline{T(B_{<1}(0))} \supset B_{<\varepsilon}(0)$ then we claim that $T(B_{<1}(0)) \supset B_{<c}(0)$.

From $\overline{T(B_{<1}(0))} \supset B_{<2c}(0)$ we deduce

$$\overline{T(B_{<\frac{1}{2^n}}(0))} \supset B_{<\frac{c}{2^{n-1}}}(0) \quad \forall n \geqslant 0.$$

Let $y \in B_{<c}(0)$ : we are going to construct a preimage $x$ of $y$ with $x \in B_{<1}(0)$.

Now $y \in B_{<c}(0) \subset \overline{T(B_{<1/2}(0))}$,

so pick $z_1 \in T(B_{<1/2}(0))$ with

$$\| y - T(z_1) \| < c/2 .$$

Now $y - T(z_1) \in B_{<c/2}(0) \subset \overline{T(B_{<1/4}(0))}$

and pick $z_2 \in B_{<1/4}(0)$ with

$$\| y - T(z_1) - T(z_2) \| < c/4 .$$

This way we construct a sequence $(z_n)_{n \geq 1}$ with

$$z_n \in B_{<1/2^n}(0) \qquad n \geq 1$$

and $$\| y - (T(z_1) + \dots + T(z_n)) \| < \frac{c}{2^n} .$$

Let $x_n := z_1 + \dots + z_n$. Since $\| z_n \| < \frac{1}{2^n}$ $x_n$ is a Cauchy sequence in $V$ and since $V$ is Banach $x := \lim_{n \to \infty} x_n$ exists.

Since $T$ is continuous and linear,

$$T(x) = y.$$

Finally:
$$\|x\| \leq \sum_{k=1}^{\infty} \|z_k\|$$

Now: $\|z_1\| < \frac{1}{2}$

and
$$\sum_{k=2}^{\infty} \|z_k\| \leq \sum_{k=2}^{\infty} \frac{1}{2^k} = \frac{1}{2}$$

and hence $\|x\| < 1$.

$\boxed{\Sigma}$

We record as an imm. consequence of Thm 4.19.

<u>Cor. 4.20.</u> Let $V, W$ be Banach spaces and $T: V \to W$ bounded bijective. Then $T^{-1}: W \to V$ is bounded.

<u>Cor. 4.21.</u> Let $V$ be a $\mathbb{K}$-v.s. endowed with two norms $\|\ \|_1, \|\ \|_2$ for which it is Banach. Assume $\exists\, c > 0 : \|v\|_2 \leq c\|v\|_1 \ \forall v \in V$. Then $\exists\, D > 0 : \|v\|_1 \leq D\|v\|_2 \ \forall v \in V$.

<u>Proof:</u> Apply Cor. 4.20 to $id : (V, \|\ \|_1) \to (V, \|\ \|_2)$.

$\boxed{\xi}$

Next:

<u>Thm. 4.22</u> (closed graph thm.) Let $V, W$ be Banach spaces and $T: V \to W$ a linear map. Assume $\text{graph}(T) = \{(v, T(v)) : v \in V\} \subset V \times W$ is closed. Then $T$ is bounded.

Remarks (1) The converse holds since $W$ is Hausdorff, indeed $\operatorname{graph}(T) \subset V \times W$ is the inverse image of the diagonal in $W \times W$ under $T \times \mathrm{Id}$.

(2) For a linear $T : V \to W$ TFAE:

(1) $\operatorname{graph}(T) \subset V \times W$ is closed.

(2) Whenever $(v_n)_{n \geq 1}$ is a sequence in $V$ with $\lim_{n \to \infty} v_n = 0$ and $\lim_{n \to \infty} T(v_n) = w$ then $w = 0$.

Indeed (1) $\Rightarrow$ (2) since

$$(0, w) = \lim_{n \to \infty} (v_n, T(v_n)) \in \operatorname{graph}(T)$$

hence $T(0) = w$.

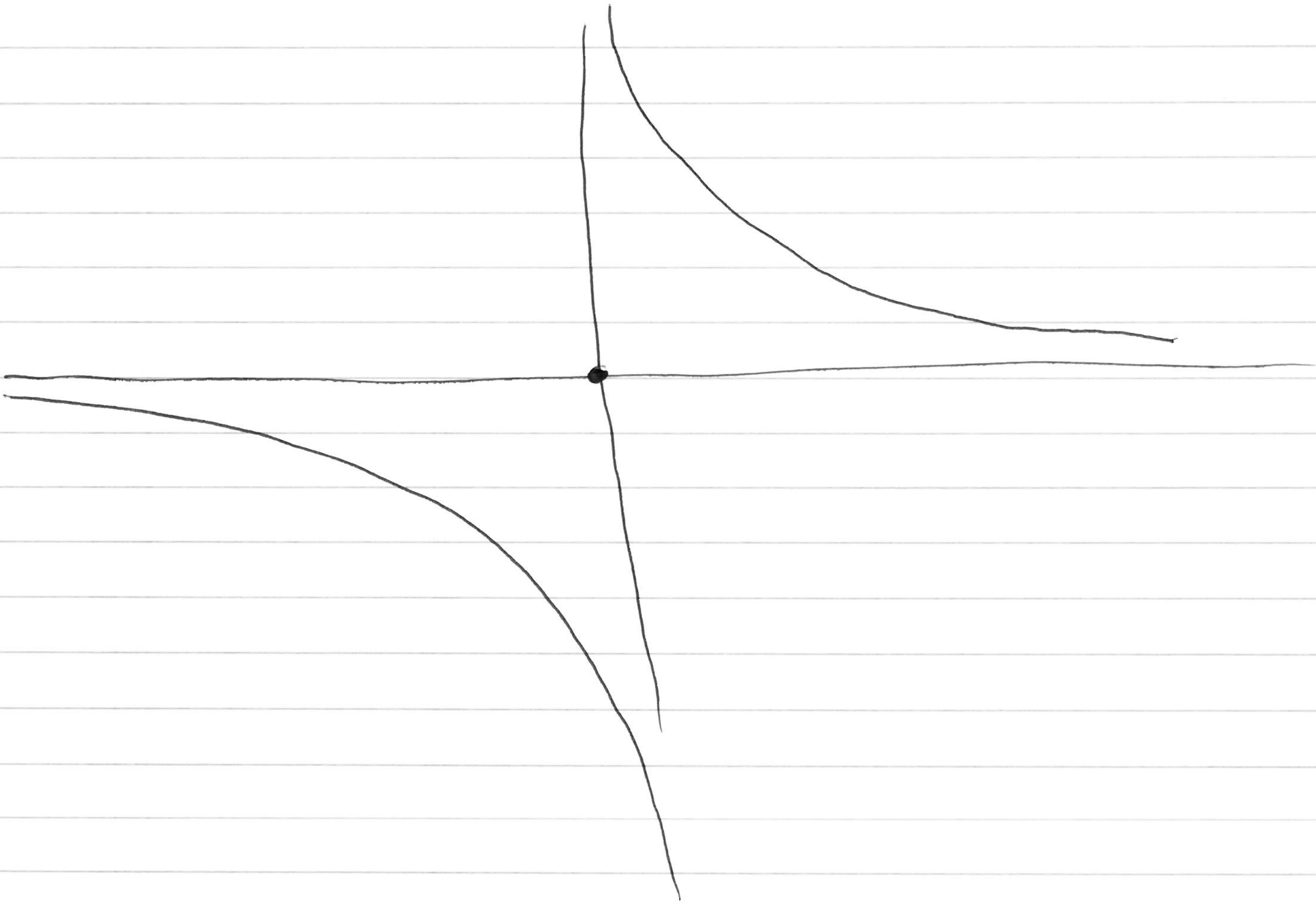
(2) $\Rightarrow$ (1): Assume $\lim_{n \to \infty} (u_n, T(u_n)) = (u, y)$

Then $v_n = u_n - u \to 0$ and $\underbrace{T(u_n) - T(u)}_{T(u_n - u)} \to y - T(u)$

$\Rightarrow y = T(u)$.

(3) $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

has closed graph but is not continuous.



Thus linearity is important.

Proof: $V \times W$ with norm $\|(v,w)\| = \|v\| + \|w\|$ is a Banach space; by assumption $\mathrm{graph}(T) \subset V \times W$ is a $\mathbb{K}$-vector subspace and is closed, hence is Banach.

Now $\mathrm{pr}_V|_{\mathrm{graph}(T)} : \mathrm{graph}(T) \longrightarrow V$ is a cont. linear bijection, hence (Cor. 4.20)

$$\left(\mathrm{pr}_V|_{\mathrm{graph}(T)}\right)^{-1} : V \longrightarrow \mathrm{graph}(T)$$ is

cont. Since $T = \mathrm{pr}_W \circ \left(\mathrm{pr}_V|_{\mathrm{graph}(T)}\right)^{-1}$

$T$ is bounded. $\boxed{\xi}$

Remark 4.24. Consider the derivative

$$D : C^1([0,1]) \longrightarrow C([0,1])$$
$$f \longmapsto f'$$

where both spaces are endowed with the sup norm

Then graph $D$ is closed : indeed if $f_n \to f \in C^1([0,1])$ uniformly and $f_n' \to g$ uniformly, Anal. 1 $\Rightarrow$ $g = f'$. But clearly $D$ is not bounded.

## 4.5. Grothendieck's Thm on closed subspaces of $L^p$.

**Thm 4.25** $(X, \mathcal{A}, \mu)$ a measure space with $\mu(X) < +\infty$. Assume

(1) $E$ is a closed subspace of $L^p(X)$, $1 \leq p < +\infty$

(2) $E \subset L^\infty(X)$

Then $E$ is finite dimensional.

**Proof**

(1) Consider the inclusion $I: E \to L^\infty(X)$, $f \mapsto f$

where $E$ is equipped with $L^p$-norm coming from $L^p(X)$. Let's verify that graph$(I)$ is closed:

Let $f_n \to f \in L^p(X)$ and $f_n \to g$ in $L^\infty(X)$.

Then $\exists (n_k)_{k \geq 1}$ such that $f_{n_k} \to f$ a.e.

But $f_{n_k} \to g$ a.e. $\Rightarrow f = g$.

Thus $I$ is bounded, $\exists\ M > 0$ s.t.

$$\|f\|_\infty \leq M \cdot \|f\|_p \quad \forall f \in E.$$

(2) $\exists\ A > 0$ with $\|f\|_\infty \leq A \|f\|_2 \quad \forall f \in E$.

Observe that since $E \subset L^\infty(X)$ and $\mu(X) < +\infty$

$E \subset L^2(X)$.

$1 \leq p \leq$ ~~$\infty$~~ $2$: Hölder inequality gives since $2/p \geq 1$:

$$\int_X |f|^p d\mu \leq \left(\int_X |f|^2 d\mu\right)^{p/2} \left(\int_X 1\, d\mu\right)^{\frac{2-p}{2}}$$

hence $\|f\|_p \leq \|f\|_2 \cdot \mu(X)^{\frac{2-p}{2p}}$

and hence $\|f\|_\infty \leq M \cdot \mu(X)^{\frac{2-p}{2p}} \|f\|_2$.

$2 < p < +\infty$ : $\quad |f(x)|^p \leqslant \|f\|_\infty^{p-2} \, |f(x)|^2$

which integrates to

$$\|f\|_p^p \leqslant \|f\|_\infty^{p-2} \, \|f\|_2^2$$

Use now $\quad \|f\|_\infty \leqslant M \|f\|_p \quad \forall f \in E$

to get $\quad \|f\|_p^p \leqslant M^{p-2} \, \|f\|_p^{p-2} \, \|f\|_2^2$

hence $\quad \|f\|_p^2 \leqslant M^{p-2} \, \|f\|_2^2$

$$\|f\|_p \leqslant M^{\frac{p-2}{2}} \, \|f\|_2 \quad \forall f \in E.$$

(3) Assume $\dim E \geqslant n$ and let $f_1, \dots, f_n$ be an orthonormal subset of $E$ (obtained by Gram-Schmidt).

Let $B = \left\{ s = (s_1, \dots, s_n) \in \mathbb{C}^n : \sum_{i=1}^{n} |s_i|^2 \leqslant 1 \right\}$

and $\forall s \in B$, $\quad f_s = \sum_{i=1}^{n} s_i f_i$.

Then $\|f_s\|_2^2 = \sum_{i=1}^{n} |s_i|^2 \leq 1$

hence $\|f_s\|_\infty \leq A \quad \forall s \in B$.

Let $\mathcal{D} \subset B$ be countable dense and

$\forall s \in \mathcal{D}$, $\quad X_s = \{ x \in X : |f_s(x)| \leq A \}$

Then $\mu(X_s) = \mu(X)$ and hence for $S = \bigcap_{s \in \mathcal{D}} X_s$

we have $\mu(S) = \mu(X)$.

Thus $\forall x \in S$ fixed, the continuous map

$$\begin{aligned} B &\longrightarrow \mathbb{C} \\ s &\longmapsto \sum_{i=1}^{n} s_i f_i(x) \end{aligned}$$

~~is $\leq A$. Hence this holds~~

is $\leq A$ on $\mathcal{D}$ hence $\leq A$ on $B$. Thus:

$\forall x \in S \ \forall s \in B : \ |f_s(x)| \leq A$.

But now $\sup_{\|s\|_2 \leq 1} |f_s(x)| = \left( \sum_{i=1}^{n} |f_i(x)|^2 \right)^{1/2}$

which implies $$\sum_{i=1}^{n} |f_i(x)|^2 \leq A^2 \quad \forall x \in S$$

Which integrates to:

$$n \leq \mu(X) A^2 .$$

Hence $\dim E \leq \mu(X) A^2 + 1$. $\boxed{\S}$