Exercise Sheet 1

Due: To be handed in before 30.09.2024 at 12:15.

1. Arc length

Let $c \in C^1([0,1], \mathbb{R}^n)$. Show that the metric definition of arc length coincides with $L(c) := \int_0^1 |c'(t)|\,\mathrm{d}t$.

2. Osculating circle

Let $c \in C^2(I, \mathbb{R}^2)$ be a curve parametrized by arc length. A circle $S \subset \mathbb{R}^2$ with center $q \in \mathbb{R}^2$ and radius $r \geq 0$ is called *osculating circle* to c at the point $t \in I$ if S coincides with c at the point $c(t)$ up to second order.

Show that if $c''(t) \neq 0$ then there is a unique osculating circle S to c at the point t. Find q, r and a parametrization α of S with $\alpha(t) = c(t), \alpha'(t) = c'(t)$ and $\alpha''(t) = c''(t)$.

3. Curvature and torsion

a) Let $c \in C^3(I, \mathbb{R}^3)$ be a Frenet curve. Show that for the curvature κ and the torsion τ of c it holds that:

$$\kappa = \frac{|c' \times c''|}{|c'|^3} \quad \text{and} \quad \tau = \frac{\det(c', c'', c''')}{|c' \times c''|^2}.$$

b) Let $r, h > 0$ and denote by σ the following reflection of $\mathbb{R}^3$:

$$\sigma : \mathbb{R}^3 \to \mathbb{R}^3,\ (x, y, z) \mapsto (x, y, -z).$$

Compute the curvature $\kappa(t)$ and the torsion $\tau(t)$ of the following Helixes:

$$\begin{aligned} c_1(t) &= \big(r\cos t, r\sin t, \frac{h}{2\pi}t\big), \\ c_2(t) &= c_1(-t), \\ c_3(t) &= \sigma \circ c_1(t). \end{aligned}$$

4. Length of curve after normal perturbation

Let $c\colon [0,l] \to \mathbb{R}^2$ $(l > 0)$ be a C^2-closed curve of constant speed one with Frenet frame (c', n). For $\delta \in \mathbb{R}$, consider the parallel curve $c_\delta\colon [0,l] \to \mathbb{R}^2$ defined by $c_\delta(s) := c(s) + \delta n(s)$ for all $s \in [0,l]$. Show that there exists an $\varepsilon > 0$ such that if $|\delta| \leq \varepsilon$, then the length $L(c_\delta)$ can be expressed solely in terms of l, δ, and the rotation index ρ_c of c.