Exercise Sheet 10

Due: To be handed in before 02.12.2024 at 12:15.

1. Embeddings

a) Find an embedding $S^k \times \mathbb{R}^l \hookrightarrow \mathbb{R}^{k+l}$, where $k, l \geq 1$.

b) Prove that if the m-dimensional manifold M is a product of spheres, then there is an embedding $M \hookrightarrow \mathbb{R}^{m+1}$.

2. The Complex Projective Space

Consider the following equivalence relation on the complex vector space $\mathbb{C}^{n+1}$:

$$x \sim y \quad \Longleftrightarrow \quad x = \lambda y \text{ for some } \lambda \in \mathbb{C} \setminus \{0\}.$$

The quotient space $\mathbb{CP}^n := \big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\sim$, equipped with the quotient topology, is called *complex projective space.*

a) Find a differentiable structure on the topological space $\mathbb{CP}^n$ such that the canonical projection

$$\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{CP}^n$$

is a differentiable map.

b) Prove that S^2 and $\mathbb{CP}^1$ are diffeomorphic.

3. Regular Values

Let M and N be manifolds of the same dimension with M compact and let $f\colon M \to N$ be a smooth map. Let $y \in N$ be a regular value of f. Prove the following statements.

a) The preimage $f^{-1}(y)$ has only finitely many elements.

b) The number of elements in the fiber over y is locally constant in N. That is, for every regular value $y \in N$ there exists a neighborhood V of y, such that all $y' \in V$ are regular values and $\#f^{-1}(y) = \#f^{-1}(y')$.

c) If the space of regular values is connected, then $\#f^{-1}(y)$ is constant for all regular values.