Exercise Sheet 11

Due: To be handed in before 09.12.2024 at 12:15.

1. Ideal Triangles in the Hyperbolic Plane

Let $H^2 \coloneqq \{x+iy \in \mathbb{C} : y > 0\}$ be the upper half-plane endowed with the hyperbolic metric $(g_{ij})(x,y) = \frac{1}{y^2}(\delta_{ij})$. The "point at infinity", ∞, denotes the "point" which corresponds to $\lim_{y\to\infty}(x,y)$ for all $x \in \mathbb{R}$. An *ideal triangle* is a geodesic triangle whose all vertices lie on the x-axis or whose two vertices lie on the x-axis and one at the point at infinity.

a) Prove that every ideal triangle is congruent to the ideal triangle with vertices $A = (0,0)$, $B = (1,0)$ and $C = \infty$.

b) Prove that the area of any ideal triangle is π.

2. Hopf Fibration

Let $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the canonical projection from Ex. 2 of Exercise Sheet 10. The *Hopf fibration*

$$H\colon S^{2n+1} \to \mathbb{CP}^n$$

is given by the restriction of π to $S^{2n+1} \subset \mathbb{C}^{n+1} \setminus \{0\}$.

a) Let $n = 1$. Describe the fibers of H over a point $x \in \mathbb{CP}^1$, that is, $H^{-1}(x)$.

b) Prove that $H\colon S^{2n+1} \to \mathbb{CP}^n$ is a submersion.

3. Mapping Degree of Gauss Map

Let $M \subset \mathbb{R}^3$ be a compact, connected surface (without boundary) with exterior Gauss map $N\colon M \to S^2$. Prove that

$$\deg(N) = \frac{1}{2}\chi(M).$$

Hint: Use Exercise 3 of Sheet 7.