Exercise Sheet 12

Due: To be handed in before 16.12.2024 at 12:15.

1. Orthogonal Structures

Let $\pi \colon E \to M$ be a vector bundle of rank k over a manifold M. An *orthogonal structure* g on E assigns to every point $p \in M$ a scalar product g_p on the fiber $E_p := \pi^{-1}(p)$, such that for all sections s, s' the map $p \mapsto g_p(s(p), s'(p))$ is smooth.

Prove that every vector bundle admits an orthogonal structure.

Hint: Use a partition of unity.

2. Line Bundles

a) Prove that every vector bundle of rank 1 over a simply connected manifold is trivial.

b) Prove that, up to isomorphism, there exist exactly two vector bundles of rank 1 over S^1.

3. F-related Vector Fields

Let $F \colon M \to N$ be a C^1-map, $X, X' \in \Gamma(TM)$ and $Y, Y' \in \Gamma(TN)$ vector fields. We say that Y is *F-related* to X if

$$Y_{F(p)} = dF_p(X_p) \text{ for every } p \in M.$$

a) Suppose that Y is F-related to X and let φ, ϕ the local flows of X and Y, respectively. Show that

$$F \circ \varphi^t = \phi^t \circ F.$$

b) Prove that if Y is F-related to X and Y' is F-related to X', then $[Y, Y']$ is F-related to $[X, X']$.