

Exercise Sheet 4

Due: To be handed in before 21.10.2024 at 12:15.

1. Tubular surface

Let $c : [0, L] \rightarrow \mathbb{R}^3$ be a smooth Frenet curve, parametrized by arc-length with normal vector n and binormal vector b . Show that if $r > 0$ is sufficiently small, then the tubular surface $f : [0, L] \times \mathbb{R} \rightarrow \mathbb{R}^3$ around c defined by

$$f(t, \varphi) := c(t) + r(\cos \varphi \cdot n(t) + \sin \varphi \cdot b(t))$$

is regular and the area of $f|_{[0, L] \times [0, 2\pi]}$ equals $2\pi rL$.

2. Torus

Let $a > r > 0$ and $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the parametrization of a torus T , defined as

$$f(x, y) := ((a + r \cos x) \cos y, (a + r \cos x) \sin y, r \sin x).$$

Prove that:

- (a) If a geodesic is at some point tangential to the circle $x = \frac{\pi}{2}$ then it must be contained in the region of T with $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
- (b) Suppose that a geodesic $c : \mathbb{R} \rightarrow T$, which crosses the circle $x = 0$ with an angle $\theta \in (0, \frac{\pi}{2})$, also intersects the circle $x = \pi$, then

$$\cos \theta \leq \frac{a - r}{a + r}.$$

3. Energy

Let $M \subset \mathbb{R}^n$ be a submanifold, $c_0 : [a, b] \rightarrow M$ a smooth curve and

$$E(c_0) := \frac{1}{2} \int_a^b |\dot{c}_0(t)|^2 dt$$

its energy.

- (a) Show that $L(c_0)^2 \leq 2(b - a)E(c_0)$ with an equality if and only if c_0 is parametrized proportionally to arc-length.
- (b) Compute $\frac{d}{ds} \Big|_{s=0} E(c_s)$ for a smooth variation $\{c_s\}_{s \in (-\varepsilon, \varepsilon)}$ of c_0 in M .
- (c) Characterize geodesics in M using the energy.