Exercise Sheet 5

Due: To be handed in before 28.10.2024 at 12:15.

1. Elliptic Points

A point $p \in M \subset \mathbb{R}^{m+1}$ on a hypersurface is called *elliptic* if the second fundamental form is (positive or negative) definite. Show that if M is compact then it has elliptic points.

2. Mean Curvature

Let $M \subset \mathbb{R}^3$ be a surface and $p \in M$ a point. Fix $0 \neq v_0 \in TM_p$. Let $H(p)$ be the mean curvature in p and denote by $\kappa_p(\theta) := h_p(v, v)$ the normal curvature in direction v, where $v \in TM_p$, $|v| = 1$, forms an angle θ with v_0.

Prove that

$$H(p) = \frac{1}{\pi} \int_0^{\pi} \kappa_p(\theta)\, d\theta.$$

3. Local Isometries

Let $f, \tilde{f} \colon \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}^3$ be two immersions, given by

$$f(x, y) := (x \sin y, x \cos y, \log x),$$
$$\tilde{f}(x, y) := (x \sin y, x \cos y, y).$$

a) Show that f and $\tilde{f}$ have the same Gauss curvature (as functions of (x, y)).

b) Are f and $\tilde{f}$ (locally) isometric?

Hint: Consider the level sets of the Gauss curvature and the curves orthogonal to these.