Exercise Sheet 7

Due: To be handed in before 11.11.2024 at 12:15.

1. Characterization of the Sphere

Prove the following lemma due to H. Hopf:

Lemma. Let $m \geq 1$, M be a compact, connected, m-dimensional submanifold of $\mathbb{R}^{m+1}$. Suppose that for each vector $v \in S^m$ there exists $\lambda = \lambda(v) \in \mathbb{R}$ such that M is symmetric with respect to reflections on the hyperplane $E_{v,\lambda} := \{x \in \mathbb{R}^{m+1} : \langle x, v \rangle = \lambda\}$, then M is a sphere.
Hint: Show first that upon translation one can arrange that M is symmetric with respect to all coordinate hyperplanes and, hence, centrally symmetric with respect to the origin.

2. Non-positively Curved Surfaces

Let $M \subset \mathbb{R}^3$ be a surface with Gauss curvature $K \leq 0$. Prove the following assertions (we assume $a < b$).

a) There is no simple geodesic loop (in particular no simple C^∞-closed geodesic) $c\colon [a, b] \to M$ whose trace bounds a topological disk in M.

b) There is no pair of injective geodesics $c_1, c_2\colon [a, b] \to M$ such that $c_1(a) = c_2(a)$ and $c_1(b) = c_2(b)$ are the only common points and the union of the traces bounds a topological disk.

c) If M is homeomorphic to a cylinder and $K < 0$, then there is no pair of simple C^∞-closed geodesics $c_1, c_2\colon [a, b] \to M$ with different traces.

3. Gauss Map of the Torus

a) Let $f\colon U \to \mathbb{R}^{m+1}$, $U \subset \mathbb{R}^m$ open, be an immersion with Gauss map $\nu\colon U \to S^m \subset \mathbb{R}^{m+1}$. Assuming that ν is an immersion, prove that
$$A(\nu) = \int_U |K| \sqrt{\det(g_{ij})}\, dx.$$

b) Let $T \subset \mathbb{R}^3$ be a torus. Describe the image of the Gauss map and prove that
$$\int_T K\, dA = 0,$$
without using the theorem of Gauss-Bonnet.