Exercise Sheet 8

Due: To be handed in before 18.11.2024 at 12:15.

1. The Brouwer Fixed Point Theorem

Brouwer's theorem states that every continuous self-map $f\colon D \to D$ of the unit ball $D := \{x \in \mathbb{R}^n : |x| \leq 1\}$ has a fixed point.

a) Let $M \subset \mathbb{R}^3$ be a surface and $\tilde{D} \subset M$ a region diffeomorphic to the disc $D := \{x \in \mathbb{R}^2 : |x| \leq 1\}$. Consider a continuous tangent vector field $X\colon \tilde{D} \to \mathbb{R}^3$ which on $\partial\tilde{D}$ is pointing outward. Show that X has zeros in the interior of $\tilde{D}$.

b) Prove the Brouwer fixed point theorem in two dimensions using part a).

2. Hyperbolic Trigonometry

Consider a geodesic triangle with angles α, β, γ at the vertices A, B, C and sides of lengths a, b, c opposite to A, B, C, respectively, in the hyperbolic plane $(H^2, g) \subset \mathbb{R}^{2,1}$. Prove the following trigonometric identities of hyperbolic geometry:

a) $\sinh c \, \sin\beta = \sinh b \, \sin\gamma$ (law of sines),

b) $\cosh c = \cosh a \, \cosh b - \sinh a \, \sinh b \, \cos\gamma$ (law of cosines),

c) $\cos\gamma = \sin\alpha \, \sin\beta \, \cosh c - \cos\alpha \, \cos\beta$ (law of cosines for angles).

Hint: Choose B in e_3 and C in the plane spanned by e_1 and e_3, then compute the coordinates of A in two different ways.

3. Circles in H^2

Consider a circle $S \subset H^2$ of radius $r > 0$ in the hyperbolic plane. Determine the length L of S, the (constant) geodesic curvature κ of S (with respect to the inward normal), and the area A of the disk $D \subset H^2$ bounded by S. (Hint: save work by making use of a relation between these three quantities.) What is the behavior of $\kappa = \kappa(r)$ when $r \to \infty$?