

Exercise Sheet 9

Due: To be handed in before 25.11.2024 at 12:15.

1. Isometries of the Hyperbolic Upper Half-Plane

Let $H^2 := \{x+iy \in \mathbb{C} : y > 0\}$ be the upper half-plane endowed with the hyperbolic metric $(g_{ij})(x, y) = \frac{1}{y^2}(\delta_{ij})$. The group $\mathrm{GL}(2, \mathbb{R})$ acts on H^2 in the following way.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$, then the group action is given by

$$\begin{aligned} z &\mapsto \frac{az+b}{cz+d}, & \text{if } ad-bc > 0, \\ z &\mapsto \frac{a\bar{z}+b}{c\bar{z}+d}, & \text{if } ad-bc < 0. \end{aligned}$$

- a) Prove that $\mathrm{GL}(2, \mathbb{R})$ acts on H^2 by isometries.
- b) Prove that the isometry group of H^2 is isomorphic to $\mathrm{PGL}(2, \mathbb{R}) := \mathrm{GL}(2, \mathbb{R})/\{\lambda I : \lambda \neq 0\}$, where I is the identity in $\mathrm{GL}(2, \mathbb{R})$.

2. Differentiable Structures on \mathbb{R}

- a) Let $\varphi, \psi: \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$\varphi(x) := x^3 \quad \text{and} \quad \psi(x) := \begin{cases} x, & x < 0, \\ 2x, & x \geq 0. \end{cases}$$

Do the atlases $\mathcal{A}_1 := \{\mathrm{id}_{\mathbb{R}}\}$, $\mathcal{A}_2 := \{\varphi\}$ and $\mathcal{A}_3 := \{\psi\}$ induce different differentiable structures on \mathbb{R} ?

- b) Prove that if two C^∞ -manifolds are homeomorphic to \mathbb{R} , then they are diffeomorphic to each other.

Hint: Prove that every differentiable structure on \mathbb{R} is diffeomorphic to the one induced by \mathcal{A}_1 .

3. Proper Functions

A function $f: M \rightarrow \mathbb{R}$ is called *proper* if $f^{-1}(K)$ is compact for every compact subset $K \subset \mathbb{R}$.

Let M be a C^∞ -manifold. Prove that there exists a proper C^∞ -function on M .

Hint: Use a partition of unity.