Solution 1

1. Arc length

Let $c \in C^1([0,1],\mathbb{R}^n)$. Show that the metric definition of arc length coincides with $L(c) := \int_0^1 |c'(t)|\, \mathrm{d}t$.

Solution:

We'll denote by $l(c)$ the length of the curve c given by the metric definition.
We first show $l(c) \le L(c)$. Let $0 = t_0 \le \ldots \le t_n = 1$ be a finite partition of $[0,1]$, then

$$\begin{aligned}\sum_{i=1}^n d(c(t_{i-1}), c(t_i)) &= \sum_{i=1}^n |c(t_i) - c(t_{i-1})| = \sum_{i=1}^n \left| \int_{t_{i-1}}^{t_i} c'(\tau)\mathrm{d}\tau \right| \\ &\le \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |c'(\tau)|\, \mathrm{d}\tau = \int_0^1 |c'(\tau)|\, \mathrm{d}\tau,\end{aligned}$$

and thus $l(c) \le L(c)$.

We now show the other inequality: let $\varepsilon > 0$ and choose $n \ge 2$ big enough such that $h = h_n := \frac{1}{n} < \varepsilon$. Consider the partition of $[0,1]$ given by $t_k := \frac{k}{n}$ for $k = 0, \ldots, n$, then

$$\begin{aligned}\frac{1}{h}\int_0^{1-h} d(c(t), c(t+h))\ \mathrm{d}t &= \frac{1}{h}\int_0^{t_{n-1}} d(c(t), c(t+h))\, \mathrm{d}t \\ &= \frac{1}{h}\sum_{k=0}^{n-2}\int_{t_k}^{t_{k+1}} d(c(t), c(t+h))\, \mathrm{d}t \\ &= \frac{1}{h}\sum_{k=0}^{n-2}\int_0^h d(c(s+t_k), c(s+t_{k+1}))\, \mathrm{d}s \\ &= \frac{1}{h}\int_0^h \sum_{k=0}^{n-2} d(c(s+t_k), c(s+t_{k+1}))\, \mathrm{d}s \\ &\le \frac{1}{h}\int_0^h l(c)\, \mathrm{d}s = l(c),\end{aligned}$$

where in the third equality we have used the substitution $s = t - t_k$. Using Fatou's lemma we obtain

$$\begin{aligned}\int_0^{1-\varepsilon} |c'(t)|\ \mathrm{d}t &= \int_0^{1-\varepsilon} \lim_{n\to\infty} \left| \frac{c(t+h_n) - c(t)}{h_n} \right|\ \mathrm{d}t \\ &\le \liminf_{n\to\infty} \frac{1}{h_n}\int_0^{1-\varepsilon} d(c(t), c(t+h_n))\, \mathrm{d}t \le l(c),\end{aligned}$$

and the statement follows by letting $\varepsilon \to 0$.

2. Osculating circle

Let $c \in C^2(I,\mathbb{R}^2)$ be a curve parametrized by arc length. A circle $S \subset \mathbb{R}^2$ with center $q \in \mathbb{R}^2$ and radius $r \ge 0$ is called *osculating circle* to c at the point $t \in I$ if S coincides with c at the point $c(t)$ up to second order.

Show that if $c''(t) \neq 0$ then there is a unique osculating circle S to c at the point t. Find q, r and a parametrization

α of S with $\alpha(t) = c(t), \alpha'(t) = c'(t)$ and $\alpha''(t) = c''(t)$.

Solution:

We start with two remarks:

- Two curves α, β coincide up to second order at t_0 if

$$\alpha(t_0) = \beta(t_0), \quad \dot{\alpha}(t_0) = \dot{\beta}(t_0), \quad \ddot{\alpha}(t_0) = \ddot{\beta}(t_0).$$

- Every regular C^2-curve $c : I \to \mathbb{R}^2$ is a Frenet curve. If c is parametrized by arc-length then

$$\begin{aligned} &e_1(t) := \dot{c}(t), \\ &e_2(t) := e_1(t) \text{ rotated } \frac{\pi}{2} \text{ to the left.} \end{aligned}$$

  From $\langle \dot{c}(t), \ddot{c}(t) \rangle = \frac{1}{2} \langle \dot{c}(t), \dot{c}(t) \rangle' = 0$ it follows that $\ddot{c}(t)$ and $e_2(t)$ are parallel and $\ddot{c}(t) = \kappa_{\text{or}}(t) \cdot e_2(t)$. Therefore (for a Frenet curve)

$$\ddot{c}(t) \neq 0 \iff \kappa_{\text{or}}(t) \neq 0.$$

We claim that the circle S with center

$$q := c(t_0) + \frac{1}{\kappa_{\text{or}}(t_0)} e_2(t_0)$$

and radius

$$r := \frac{1}{|\kappa_{\text{or}}(t_0)|}$$

is the unique osculating circle for c at t_0.

We parametrize S as follows

$$\alpha(t) = q + \frac{1}{\kappa_{\text{or}}(t_0)} \Big(\sin\big(\kappa_{\text{or}}(t_0)(t - t_0)\big) \cdot e_1(t_0) - \cos\big(\kappa_{\text{or}}(t_0)(t - t_0)\big) \cdot e_2(t_0) \Big).$$

Then

$$\begin{aligned} \alpha'(t) &= \cos\big(\kappa_{\text{or}}(t_0)(t - t_0)\big) \cdot e_1(t_0) + \sin\big(\kappa_{\text{or}}(t_0)(t - t_0)\big) \cdot e_2(t_0), \\ \alpha''(t) &= \kappa_{\text{or}}(t_0) \Big(- \sin\big(\kappa_{\text{or}}(t_0)(t - t_0)\big) \cdot e_1(t_0) + \cos\big(\kappa_{\text{or}}(t_0)(t - t_0)\big) \cdot e_2(t_0) \Big). \end{aligned}$$

At $t = t_0$ we have

$$\begin{aligned} \alpha(t_0) &= q - \frac{1}{\kappa_{\text{or}}(t_0)} \cdot e_2(t_0) = c(t_0), \\ \dot{\alpha}(t_0) &= e_1(t_0) = \dot{c}(t_0), \\ \ddot{\alpha}(t_0) &= \kappa_{\text{or}}(t_0) \cdot e_2(t_0) = \ddot{c}(t_0), \end{aligned}$$

and so S is an osculating circle for c at t_0.

We now prove uniqueness. Let T be another osculating circle for c at t_0 and denote by β an arc-length parametrization of T (with $\beta(t_0) = c(t_0), \dot{\beta}(t_0) = \dot{c}(t_0)$ and $\ddot{\beta}(t_0) = \ddot{c}(t_0)$). Let a_1, a_2 be a Frenet frame for α and b_1, b_2 a Frenet frame for β, then

$$\beta(t_0) = c(t_0) = \alpha(t_0)$$

and

$$b_1(t_0) = \dot{\beta}(t_0) = \dot{c}(t_0) = \dot{\alpha}(t_0) = a_1(t_0)$$

so also $b_2(t_0) = a_2(t_0)$.

Moreover

$$\kappa_{\mathrm{or},\beta}(t_0) \cdot b_2(t_0) = \ddot{\beta}(t_0) = \ddot{c}(t_0) = \ddot{\alpha}(t_0) = \kappa_{\mathrm{or},\alpha}(t_0) \cdot a_2(t_0),$$

and hence $\kappa_{\mathrm{or},\beta}(t_0) = \kappa_{\mathrm{or},\alpha}(t_0)$. Notice that circles have constant curvature κ, that is $\kappa(t_0) = \kappa(t)$, hence $\kappa_{\mathrm{or},\alpha}(t) = \kappa_{\mathrm{or},\beta}(t)$ for all t. It follows directly from the Fundamental Theorem of local curve theory that $\alpha(t) = \beta(t)$ and therefore $S = T$.

3. Curvature and torsion

a) Let $c \in C^3(I, \mathbb{R}^3)$ be a Frenet curve. Show that for the curvature κ and the torsion τ of c it holds that:

$$\kappa = \frac{|c' \times c''|}{|c'|^3} \quad \text{and} \quad \tau = \frac{\det(c', c'', c''')}{|c' \times c''|^2}.$$

b) Let $r, h > 0$ and denote by σ the following reflection of $\mathbb{R}^3$:

$$\sigma : \mathbb{R}^3 \to \mathbb{R}^3,\ (x, y, z) \mapsto (x, y, -z).$$

Compute the curvature $\kappa(t)$ and the torsion $\tau(t)$ of the following Helixes:

$$\begin{aligned} c_1(t) &= \big(r \cos t, r \sin t, \frac{h}{2\pi} t\big), \\ c_2(t) &= c_1(-t), \\ c_3(t) &= \sigma \circ c_1(t). \end{aligned}$$

Solution:

a) From $e_1 = \frac{\dot{c}}{|\dot{c}|}$ it follows that $\dot{c} = |\dot{c}| \cdot e_1$ and from the first Frenet equation we have $\dot{e}_1 = |\dot{c}|\kappa \cdot e_2$, so

$$\ddot{c} = (|\dot{c}| \cdot e_1)' = |\dot{c}|' \cdot e_1 + |\dot{c}| \cdot \dot{e}_1 = |\dot{c}|' \cdot e_1 + |\dot{c}|^2 \kappa \cdot e_2, \tag{0..1}$$

and

$$\begin{aligned} \dot{c} \times \ddot{c} &= |\dot{c}|' \cdot \dot{c} \times e_1 + |\dot{c}|^2 \kappa \cdot \dot{c} \times e_2 \\ &= |\dot{c}|' \, |\dot{c}| \cdot e_1 \times e_1 + |\dot{c}|^3 \kappa \cdot e_1 \times e_2 \\ &= |\dot{c}|^3 \kappa \cdot e_1 \times e_2 \\ &= |\dot{c}|^3 \kappa \cdot e_3, \end{aligned}$$

thus $|\dot{c} \times \ddot{c}| = |\dot{c}|^3 |\kappa| = |\dot{c}|^3 \kappa$ since the i-th Frenet curvature is positive for any $1 \le i \le n-2$ and any Frenet curve in $\mathbb{R}^n$. Then we have

$$\kappa = \frac{|\dot{c} \times \ddot{c}|}{|\dot{c}|^3}.$$

Moreover, using (0..1) and the Frenet equation for $\dot{e}_2$ we obtain

$$\begin{aligned}\dddot{c} &= |\dot c|'' \cdot e_1 + |\dot c|' \cdot \dot e_1 + \left(|\dot c|^2\kappa\right)' \cdot e_2 + |\dot c|^2\kappa \cdot \dot e_2 \\ &= |\dot c|'' \cdot e_1 + \left(|\dot c|'|\dot c|\kappa + (|\dot c|^2\kappa)'\right) \cdot e_2 + |\dot c|^2\kappa \cdot \dot e_2 \\ &= |\dot c|'' \cdot e_1 + \left(|\dot c|'|\dot c|\kappa + (|\dot c|^2\kappa)'\right) \cdot e_2 + |\dot c|^2\kappa\left(-|\dot c|\kappa \cdot e_1 + |\dot c|\tau \cdot e_3\right) \\ &= \underbrace{\left(|\dot c|'' - |\dot c|^3\kappa^2\right)}_{=:A} \cdot e_1 + \underbrace{\left(|\dot c|'|\dot c|\kappa + (|\dot c|^2\kappa)'\right)}_{=:B} \cdot e_2 + \underbrace{|\dot c|^3\kappa\tau}_{=:C} \cdot e_3.\end{aligned}$$

Consequently we can compute $\det(\dot c, \ddot c, \dddot c)$ as follows:

$$\begin{aligned}\det(\dot c, \ddot c, \dddot c) &= \det\left(|\dot c| \cdot e_1, |\dot c|' \cdot e_1 + |\dot c|^2\kappa \cdot e_2, A \cdot e_1 + B \cdot e_2 + C \cdot e_3\right) \\ &= \det\left(|\dot c| \cdot e_1, |\dot c|^2\kappa \cdot e_2, |\dot c|^3\kappa\tau \cdot e_3\right) \\ &= |\dot c|^6\kappa^2\tau \det\left(e_1, e_2, e_3\right) \\ &= |\dot c|^6\kappa^2\tau \\ &= \tau \cdot |\dot c \times \ddot c|^2,\end{aligned}$$

which proves the statement.

b) We'll denote by $\kappa_1, \kappa_2, \kappa_3$ and τ_1, τ_2, τ_3 curvature and torsion of the curves c_1, c_2 and c_3, respectively. We compute

$$\begin{aligned}c_1(t) &= (r\cos t, r\sin t, \frac{h}{2\pi}t), \\ \dot c_1(t) &= \big(-r\sin t, r\cos t, \frac{h}{2\pi}\big), \\ \ddot c_1(t) &= (-r\cos t, -r\sin t, 0), \\ \dddot c_1(t) &= (r\sin t, -r\cos t, 0).\end{aligned}$$

It holds that

$$\begin{aligned}\dot c_1 \times \ddot c_1 &= \big(r\frac{h}{2\pi}\sin t, -r\frac{h}{2\pi}\cos t, r^2\big), \\ |\dot c_1 \times \ddot c_1| &= \big(r^2\frac{h^2}{4\pi^2} + r^4\big)^{\frac12} = r\big(\frac{h^2}{4\pi^2} + r^2\big)^{\frac12}, \\ |\dot c_1| &= \big(r^2 + \frac{h^2}{4\pi^2}\big)^{\frac12},\end{aligned}$$

and therefore

$$\kappa_1 = \frac{|\dot c_1 \times \ddot c_1|}{|\dot c_1|^3} = \frac{r}{r^2 + \frac{h^2}{4\pi^2}}.$$

With

$$\begin{aligned}\det(\dot c_1, \ddot c_1, \dddot c_1) &= \det\begin{pmatrix} -r\sin t & -r\cos t & r\sin t \\ r\cos t & -r\sin t & -r\cos t \\ \frac{h}{2\pi} & 0 & 0\end{pmatrix} \\ &= r^2\frac{h}{2\pi}\cos^2 t + r^2\frac{h}{2\pi}\sin^2 t = r^2\frac{h}{2\pi},\end{aligned}$$

it follows that

$$\tau_1 = \frac{\det\left(\dot c_1, \ddot c_1, \dddot c_1\right)}{|\dot c_1 \times \ddot c_1|^2} = \frac{r^2\frac{h}{2\pi}}{r^2\left(\frac{h^2}{4\pi^2} + r^2\right)} = \frac{\frac{h}{2\pi}}{\frac{h^2}{4\pi^2} + r^2}.$$

For c_2 we have

$$\begin{aligned} c_2(t) &= c_1(-t), \\ \dot{c}_2(t) &= -\dot{c}_1(-t), \\ \ddot{c}_2(t) &= \ddot{c}_1(-t), \\ \dddot{c}_2(t) &= -\dddot{c}_1(-t), \end{aligned}$$

therefore

$$\kappa_2(t) = \frac{|\dot{c}_2(t) \times \ddot{c}_2(t)|}{|\dot{c}_2(t)|^3} = \frac{|-\dot{c}_1(-t) \times \ddot{c}_1(-t)|}{|-\dot{c}_1(-t)|^3} = \kappa_1(-t) = \frac{r}{r^2 + \frac{h^2}{4\pi^2}},$$

and

$$\begin{aligned} \tau_2(t) &= \frac{\det(\dot{c}_2(t), \ddot{c}_2(t), \dddot{c}_2(t))}{|\dot{c}_2(t) \times \ddot{c}_2(t)|^2} \\ &= \frac{\det(-\dot{c}_1(-t), \ddot{c}_1(-t), -\dddot{c}_1(-t))}{|-\dot{c}_1(-t) \times \ddot{c}_1(-t)|^2} = \tau_1(-t) = \frac{\frac{h}{2\pi}}{\frac{h^2}{4\pi^2} + r^2}. \end{aligned}$$

Finally, note $\dot{c}_3 = \sigma(\dot{c}_1)$, $\ddot{c}_3 = \sigma(\ddot{c}_1)$, $\dddot{c}_3 = \sigma(\dddot{c}_1)$, we have $|\dot{c}_3 \times \ddot{c}_3| = |\sigma(\dot{c}_1) \times \sigma(\ddot{c}_1)| = |\dot{c}_1 \times \ddot{c}_1|$ and $\det(\dot{c}_3, \ddot{c}_3, \dddot{c}_3) = \det(\sigma)\det(\dot{c}_1, \ddot{c}_1, \dddot{c}_1) = -\det(\dot{c}_1, \ddot{c}_1, \dddot{c}_1)$, so

$$\begin{aligned} \kappa_3 = \kappa_1 &= \frac{r}{r^2 + \frac{h^2}{4\pi^2}}, \\ \tau_3 = -\tau_1 &= -\frac{\frac{h}{2\pi}}{\frac{h^2}{4\pi^2} + r^2}. \end{aligned}$$

4. Length of curve after normal perturbation

Let $c\colon [0,l] \to \mathbb{R}^2$ ($l > 0$) be a C^2-closed curve of constant speed one with Frenet frame (c', n). For $\delta \in \mathbb{R}$, consider the parallel curve $c_\delta\colon [0,l] \to \mathbb{R}^2$ defined by $c_\delta(s) := c(s) + \delta n(s)$ for all $s \in [0,l]$. Show that there exists an $\varepsilon > 0$ such that if $|\delta| \le \varepsilon$, then the length $L(c_\delta)$ can be expressed solely in terms of l, δ, and the rotation index ρ_c of c.

Solution:

Let κ_{or} be the oriented curvature of c, and recall that

$$\int_0^l \kappa_{\mathrm{or}}(s)\, ds = \int_0^l \theta'(s)\, ds = \theta(l) - \theta(0) = 2\pi\rho_c$$

for any continuous function $\theta\colon [0,l] \to \mathbb{R}$ such that $c' = (\cos(\theta), \sin(\theta))$. By the second Frenet equation, $n' = -\kappa_{\mathrm{or}} c'$, thus

$$c_\delta' = c' + \delta n' = (1 - \delta\kappa_{\mathrm{or}})c'.$$

Since c has unit speed, $|c_\delta'| = |1 - \delta\kappa_{\mathrm{or}}|$. Let

$$\varepsilon := \inf\{|\kappa_{\mathrm{or}}(s)|^{-1} : s \in [0,l],\, \kappa_{\mathrm{or}}(s) \neq 0\},$$

and note that $\varepsilon \in (0,\infty)$ by continuity of κ_{or} on $[0,l]$ and since $\kappa_{\mathrm{or}} \not\equiv 0$ (ε is the minimal curvature radius of c). If $|\delta| \leq \varepsilon$, then $|c'_\delta| = 1 - \delta\kappa_{\mathrm{or}}$ and thus

$$L(c_\delta) = \int_0^l 1 - \delta\kappa_{\mathrm{or}}(s)\, ds = l - 2\pi\delta\rho_c.$$